BPS COUNTING ON SINGULAR VARIETIES

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Abstract. We define new partition functions for theories with targets on toric singularities via products of old partition functions on crepant resolutions. We compute explicit examples and show that the new partition functions turn out to be homogeneous on MacMahon factors.

Contents

1. Motivation for counting BPS states

BPS states are special states of supersymmetric theories, with minimal energy. BPS states have had a crucial role in establishing various duality symmetries of Superstring theory. One of the reasons for their pivotal role in studying dualities stems from the availability of information on exact masses and degeneracies of these states. Degeneracy of such states in a certain supersymmetric theory is obtained from the partition function of the theory. The degeneracies depend on the background geometry. As the moduli of the background is varied, the number of states can jump across a wall of marginal stability. In other words, across such a wall a BPS state may disappear, or 'decay', giving rise to a different spectrum of BPS states. The original BPS state is thus stable on one side of the wall, while the decay products are stable on the other. Indeed, when D-branes are realized as BPS states, they are defined by the stable BPS states only. The D-brane spectrum thus changes across walls in the moduli space. Characterising the jumps of degeneracy of BPS states across walls in the moduli space, notwithstanding the continuity of appropriate correlation functions, has been of immense interest recently. These studies unearthed a rich mathematical structure within the scope of topological string theories.

A class of BPS states in topological string theories is furnished by D-branes wrapping homology cycles of the target space. These D-branes as well as their bound states are described as objects in the derived category of coherent sheaves of the target space or objects in the Fukaya category, within the scope of the topological B or A models, respectively. On a Calabi–Yau target the walls of marginal stability are detected via the alignment of the charges of the D-branes in the spectrum. Across a wall a D-brane decays into a finite or infinite collection of branes, with the charge of the parent brane aligning with the totality of charges of the products on the wall.

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Now, the partition function of the A-model generates the Gromov–Witten invariants of Calabi–Yau threefolds from the world-sheet perspective. From the target space perspective, they count the Gopakumar–Vafa invariants. The GW invariants count holomorphic curves on the threefold, whereas the GV invariants count BPS states of spinning black holes in 5 dimensions obtained from M2-branes in M-theory on the Calabi–Yau threefold [AOVY]. Considering the topological A-model on the target

$$\mathbb{R}^3 \times X \times S^1 \ ,$$

where X denotes the Calabi–Yau space without four-cycles and S^1 designates the compact Euclidean temporal direction, the partition function also counts the number of D0- and D2-brane bound states on a single D6-brane wrapped on X. (In principle, M5-branes wrapping four-cycles in X can form bound states with M2-branes; these complications do not arise in the absence of four-cycles in X [AOVY].) The partition function is the generating function of the Donaldson–Thomas invariants. Thus the study of the degeneracy of states relates the GW, GV and DT invariants.

As discussed above, the D-brane spectrum changes from one chamber to another on the Calabi–Yau space, partitioned by the walls of marginal stability. Thus, the computation of degeneracy depends on the stability criteria of D-branes on a Calabi–Yau space. For the case of a singular variety, instead of considering different stability conditions, which pertain to the desingularization of the variety, we consider all crepant resolutions at once. We construct new partition functions for the singular space in terms of known partition functions of the crepant resolutions. This yields a definition of the partition function of the topological A-model and thence a counting of DT invariants for the singular Calabi–Yau variety.

2. New partition function and results

Suppose we have a singular Calabi–Yau space X and a finite collection of crepant resolutions $X^t \to X$ for index $t \in \mathcal{T}$, $|\mathcal{T}| < \infty$. Assume further that we have a partition function $Z_{\mathrm{old}}(Y; Q, \dots)$ defined for a smooth Calabi–Yau space Y, where $Q = (Q_1, Q_2, \dots)$ are formal variables corresponding to a basis of $H_2(Y;\mathbb{Z})$. Finally, we suppose that $H_2(X^s;\mathbb{Z}) \cong H_2(X^t;\mathbb{Z})$ for all $s, t \in \mathcal{T}$.

We define a new partition function Z_{new} for X as follows. Firstly, we identify the formal variables Q among all the resolutions. That is, we set

$$Q_i^s = Q_i^t =: Q_i \quad \text{for all} \quad s, t \in \mathcal{T} \ .$$

Secondly, we define

$$Z_{\mathrm{new}}(X; Q, \dots) := \prod_{t \in \mathcal{T}} Z_{\mathrm{old}}(X^t; Q^t, \dots) \ .$$

The new partition function captures the information from all possible resolutions of X, and thus can be regarded as a property of the singular space X itself. We consider only full resolutions, partial resolutions produce messier computations with no apparent extra information.

We can apply this approach to a number of different partition functions. In this paper, we consider the ones of curve-counting type such as the Gromov–Witten and the Donaldson–Thomas partition functions, and we obtain the following results:

Theorem. *Let X be a toric Calabi–Yau threefold without compact 4-cycles and without contractible curves, and let $Z(Y; q, Q)$ be any partition function of curve-counting type (def 5.6). Then the total partition function*

$$Z_{\mathrm{tot}}(X; q, Q) := \prod_{Y \to X} Z(Y; q, Q) \ ,$$

where the product ranges over all crepant resolutions of X, is homogeneous (def 5.2) of degree

$$d = \frac{(m^2 - m + n^2 - n - 2mn)(m + n - 2)!}{m!n!}$$

where m and n are the exponents appearing in the equation $xy - z^m u^n = 0$ that defines X.

We perform the curve-counting when the Calabi–Yau threefold is allowed to have contractible curves as well (Corollary 5.9) in particular obtaining a counting of BPS states via the topological string partition function (Corollary 5.11).

3. The mathematics of curve counting

3.1. **Gromov–Witten theory.**

Definition 3.1. By a *curve* we mean a reduced scheme C of pure dimension one. The *genus* of C is $g(C) := h^1(C;\ \mathcal{O}_C)$.

Corollary 3.2. *A connected curve C of genus 0 is a tree of rational curves.*

Definition 3.3. An n-pointed curve $\big(C; P_1, \dots, P_n\big)$ is called *prestable* if every point of C is either smooth or a node singularity and the points $P_1, \dots, P_n$ are smooth. A map $f\colon C \to X$ is called *stable* if $\big(C; P_1, \dots, P_n\big)$ is prestable and there are at least three marked or singular points on each contracted component.

Remark 3.4. Stability entails that the map f has no first-order infinitesimal deformations.

We write $\overline{\mathcal{M}}_{g,n}(X, \beta)$ for the collection of maps from stable, n-pointed curves of genus g into X for which

$$[f(C)] = f_*[C] = \beta \in H_2(X;\ \mathbb{Z}) \ .$$

Behrend and Fantechi [BF1] showed that this has a coarse moduli (Deligne–Mumford) stack, Vistoly [V] studied the intersection theory on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ and constructed a perfect obstruction theory, and [BF1] showed that there exists a virtual fundamental class of virtual dimension

$$\mathrm{vd} = (1-g)(\dim X - 3) - K_X(\beta) + n \ .$$

(We assume that X does in fact *have* a canonical class $K_X \in H^2(X;\ \mathbb{Z})$, e.g. if X is smooth.) Consequently, dimension of the classes $[\ \overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}}$ is independent of β when $K_X = 0$, that is, when X is Calabi–Yau. Moreover, the unpointed moduli $\overline{\mathcal{M}}_{0,0}(X, \beta)$ has virtual dimension zero for all g if $\dim X = 3$, so on a three-dimensional Calabi–Yau, $\overline{\mathcal{M}}_{0,0}(X, \beta)$ really “counts curves”.

Definition 3.5. Assume that $g(C) = 0$. Let

$$\mathrm{ev}_i\colon\ \overline{\mathcal{M}}_{0,n}(X, \beta) \to X \ , \quad \big(f\colon (C; P_1, \dots, P_n) \to X\big) \mapsto f(P_i)$$

be the i^{th} evaluation map. Assume that $\sum_{i=1}^n \deg(\gamma_i) = \mathrm{vd}$ for some $\gamma_i \in H^*(\ \overline{\mathcal{M}}_{0,n}(X, \beta))$. Then the *genus-*$0$ *Gromov–Witten invariants* are

$$\langle \gamma_1, \dots, \gamma_n \rangle_\beta := \mathrm{ev}_1^*(\gamma_1) \cup \dots \cup \mathrm{ev}_n^*(\gamma_n) \cap [\ \overline{\mathcal{M}}_{0,n}(X, \beta)]^{\mathrm{vir}} \ .$$

For higher genera, the definition of the Gromov–Witten invariants requires the introduction of additional data, called *descendent fields*. Since we require only genus 0 for our purposes, we refer the interested reader to [MNOP2, § 2].

When $\dim X = 3$, X is Calabi–Yau (i.e. $K_X = 0$), arbitrary genus g and $n = 0$, we have the *unmarked Gromov–Witten invariants*

$$N_{g,\beta}(X) := \int_{[\ \overline{\mathcal{M}}_{g,0}(X,\beta)]^{\mathrm{vir}}} 1 \ .$$

Example 3.6. If $X = \{\text{pt.}\}$, then $\overline{\mathcal{M}}_{g,n}(X, \beta) = \overline{\mathcal{M}}_{g,n}$, the moduli of n-pointed curves.

Example 3.7. For $X = \mathbb{P}^1$, the genus-0 Gromov–Witten invariants are just the Hurwitz numbers.

The (unmarked) Gromov–Witten invariants are usually assembled into an unreduced and a reduced generating function, respectively

$$\begin{aligned} F(X; u, v) &= \sum_\beta \sum_{g \geq 0} N_{g,\beta}(X) u^{2g-2} v^\beta \ , \text{ and} \\ F'(X; u, v) &= \sum_{\beta \neq 0} \sum_{g \geq 0} N_{g,\beta}(X) u^{2g-2} v^\beta \ , \end{aligned}$$

where $v = (v_1, \dots, v_r)$ is an appropriate vector that can be paired with the r generators of $H_2(X;\ \mathbb{Z})$. The unreduced and reduced *Gromov–Witten partition functions* are, respectively,

$$\begin{aligned} Z_{\mathrm{GW}}(X; u, v) &= \exp F(X; u, v) = 1 + \sum_\beta Z_{\mathrm{GW}}(X; u)_\beta v^\beta \ , \text{ and} \\ Z'_{\mathrm{GW}}(X; u, v) &= \exp F'(X; u, v) = 1 + \sum_{\beta \neq 0} Z'_{\mathrm{GW}}(X; u)_\beta v^\beta \ , \end{aligned}$$

where the last expressions define the homogeneous terms $Z(X;u)_\beta$ and $Z'(X;u)_\beta$ of degree β.

3.2. **Donaldson–Thomas theory.** An *ideal subsheaf* of $\mathcal{O}_X$ is a sheaf $\mathcal{I}$ such that $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_X(U)$ for each open set $U \subseteq X$. Alternatively, it is a torsion-free rank-1 sheaf with trivial determinant. It follows that $\mathcal{I}^{\vee\vee} \cong \mathcal{O}_X$. Thus the evaluation map determines a quotient

$$0 \longrightarrow \mathcal{I} \xrightarrow{\text{ev}} \mathcal{I}^{\vee\vee} \cong \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I}\mathcal{O}_X = \imath_*\mathcal{O}_Y \longrightarrow 0 \ , \tag{3.1}$$

where $Y \subseteq X$ is the support of the quotient and $\mathcal{O}_Y := (\mathcal{O}_X/\mathcal{I}\mathcal{O}_X)|_Y$ is the structure sheaf of the corresponding subspace. Let $[Y] \in H_2(X;\ \mathbb{Z})$ denote the cycle class determined by the 1-dimensional components of Y with their intrinsic multiplicities. We denote by

$$I_n(X,\beta)$$

the Hilbert scheme of ideal sheaves $\mathcal{I} \subset \mathcal{O}_X$ for which the quotient Y in (3.1) has dimension at most 1, $\chi(\mathcal{O}_Y) = n$ and $[Y] = \beta \in H_2(X;\ \mathbb{Z})$.

The work of Donaldson and Thomas was to show that $I_n(X,\beta)$ has a canonical perfect obstruction theory (originally when X is smooth, projective and $-K_X$ has non-zero sections) and a virtual fundamental class $[I_n(X,\beta)]^{\mathrm{vir}}$ of virtual dimension $\int_\beta c_1(T_X) = -K_X(\beta)$. If X is a smooth, projective Calabi–Yau threefold, then the virtual dimension is zero, and we write

$$\widetilde{N}_{n,\beta}(X) := \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1$$

for the "number" of such ideal sheaves. We assemble these numbers into an (unreduced) partition function,

$$Z_{\mathrm{DT}}(X;q,v) = \sum_\beta \sum_{n\in\mathbb{Z}} \widetilde{N}_{n,\beta}(X) q^n v^\beta = \sum_\beta Z_{\mathrm{DT}}(X;q)_\beta v^\beta \ ,$$

where again the last expression defines the unreduced terms of degree β. The degree-0 term

$$Z_{\mathrm{DT}}(X;q)_0 = \sum_{n\geq 0} \widetilde{N}_{n,0}(X) q^n$$

is of special importance: We define the *reduced* DT partition function as

$$Z'_{\mathrm{DT}}(X;q,v) = Z_{\mathrm{DT}}(X;q,v)/Z_{\mathrm{DT}}(X;q)_0 = 1 + \sum_{\beta\neq 0} Z'_{\mathrm{DT}}(X;q)_\beta v^\beta \ ,$$

once again defining the reduced terms $Z'_{\mathrm{DT}}(X;q)_\beta$ of degree β implicitly.

3.3. **The MNOP Conjecture.** For a smooth Calabi–Yau threefold X, the following relation was conjectured by MNOP:

$$Z'_{\mathrm{GW}}(X;u,v) = Z'_{\mathrm{DT}}(X;-e^{iu},v) \ .$$

It was proved in [BF2] and [L] in the compact case, and in the the case when X is a toric (and hence necessarily non-compact) Calabi–Yau threefold in [MNOP1, MNOP2]. Therefore, Gromov–Witten and Donaldson–Thomas theories provide equivalent information for Calabi–Yau threefolds.

We will pursue to illustrate our new approach using the Donaldson–Thomas partition function, for which toric computational techniques have been developed by [LLLZ]. The spaces which we consider are those toric threefolds X whose crepant resolutions have no compact 4-cycles, which implies that X is either a quotient of $\mathbb{C}^3$ or a quotient of the conifold.

4. Generalised conifolds

Given a pair of nonnegative integers n, m we consider the toric varieties

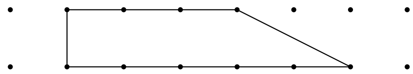
$$C_{m,n} := \bigl\{xy - z^m w^n = 0\bigr\} \subset \mathbb{C}^4 = \operatorname{Spec} \mathbb{C}[x,y,z,w] \ .$$

There are two cases: $\imath$) When $n > m = 0$, these are quotients of $\mathbb{C}^3$ by the action of $\mathbb{Z}/n$ given by $(a,b,c) \mapsto (\varepsilon a, \varepsilon^{-1} b, c)$, where $\varepsilon^n = 1$. Note that these spaces have 1-dimensional singularities, as $C_{0,n} \cong K_n \times \mathbb{C}$, where K_n is the Kleinian surface singularity $\{xy - z^n = 0\}$.

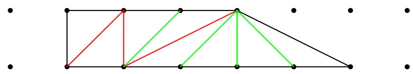
$\imath\imath$) When $n \geq m \geq 1$. The space $C_{1,1}$ is the *conifold* given by $\{xy - zw = 0\} \subset \mathbb{C}^4$. It is the standard example of an isolated 3-dimensional hypersurface singularity that is not a quotient singularity. All other spaces $C_{m,n}$ are quotients of the conifold.

The toric fan of $C_{m,n}$ is generated by a single 3-dimensional cone with ray generators, say, $(0,0,1)$, $(0,1,1)$, $(n,0,1)$ and $(m,1,1)$. As all of the ray generators lie in the $\{z = 1\}$-plane, the

the canonical divisor is trivial and the varieties are Calabi–Yau. When we refer to the toric diagram, we will henceforth only ever use the intersection with this plane, and the diagram describing $C_{m,n}$ is the strip with vertices $(0,0)$, $(0,1)$, $(n,0)$ and $(1,m)$:



If we seek to desingularise these varieties, we find that blowing up the singular locus in general results in a non-Calabi–Yau variety. This can be easily seen by constructing the star subdivision of the singular subcone and observing that the new ray generator will not lie in the $\{z=1\}$-hyperplane. However, small resolutions are crepant and therefore result in a smooth Calabi–Yau variety. We obtain these resolutions explicitly by constructing a lattice triangulation of the strip:



The internal edges in the triangulation of the strip correspond to 2-dimensional cones in the toric fan of the resolved threefold; they describe the irreducible components of the exceptional curve. Observe that the resolution has no compact 4-cycles. Its second homology is generated by the components of the exceptional curve. Each prime component of the exceptional set is a smooth rational curve.

We want to consider all possible crepant resolutions of $C_{m,n}$, which correspond to all maximal lattice triangulations of the strip (i.e. triangulations in which each triangle has area $\frac{1}{2}$). We will abuse notation and use $C_{m,n}$ to refer to the strip as well as to the variety which it defines. We collect some combinatorial properties of these triangulations.

Proposition 4.1.

(1) *There are* $\binom{m+n}{m}$ *triangulations of* $C_{m,n}$.
(2) *Each triangulation has* $m+n-1$ *interior edges and* $m+n$ *triangles.*
(3) *The Euler characteristic of any crepant resolution of* $C_{m,n}$ *is* $m+n$.

Proof. Statement (2) is obvious by induction, starting at $m=0$, $n=1$ and by observing that each triangulation of $C_{m,n}$ can be built from $C_{0,1}$ by successively adding triangles. Statement (3) follows from (2) since if X is any crepant resolution of $C_{m,n}$, then $\chi(X)=h^0(X;\mathbb{Z})+h^2(X;\mathbb{Z})$, since there are no compact 4- or 6-cycles.

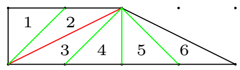
Statement (1) is illuminating to prove in two ways. First, we can proceed by induction. Suppose we pass from m to $m+1$. A triangulation of $C_{m+1,n}$ falls into one of two categories: Either there is a triangle $(m,1)$—$(m+1,1)$—$(n,0)$, and its complement is $C_{m,n}$, or there is a triangle $(m+1,1)$—$(n-1,0)$—$(n,0)$, and the complement is $C_{m+1,n-1}$. Induction gives the familiar recursion relation for the binomial coefficients, $\binom{m+n+1}{m+1}=\binom{m+n}{m}+\binom{m+n}{m+1}$.

The second method is to construct an enumeration of the triangulations directly. We claim that a triangulation corresponds one-to-one to a choice of m triangles out of $m+n$ triangles, giving the desired count. The correspondence is as follows: There is an obvious ordering of the triangles "from left to right", starting with the unique triangle t_1 that meets the edge $(0,0)$—$(0,1)$ and moving right across the unique other non-horizontal edge and arriving at the unique triangle t_{m+n} that meets the edge $(m,1)$—$(n,0)$. Each triangle has a unique horizontal edge. which is either at the *top* or at the *bottom* of the strip, corresponding to vertical coordinate 1 or 0, respectively. Precisely m triangles are at the top, and specifying which m triangles are at the top fixes the triangulation uniquely. □

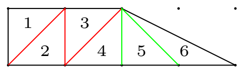
4.1. **Enumerating triangulations.** We will use the method of enumeration that was constructed in the proof of proposition 4.1. That is, we denote each triangulation of $C_{m,n}$ by a subset $T\subset\{1,2,\ldots,m+n\}$ with $|T|=m$, where the triangles t_k, $k\in T$, are at the top of the strip and $\{t_1,\ldots,t_{m+n}\}$ denotes the set of all triangles.

Example 4.2. Let $m=2$ and $n=4$. Here are some examples of triangulations of $C_{2,4}$ given by subsets of size 2 of $\{1,\ldots,6\}$.

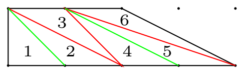
$\{1,2\}$



$\{1,3\}$



$\{3,6\}$



We also require a labelling of the interior edges. We define

$$e_i := t_i \cap t_{i+1} \ , \ i = 1, \ldots, m+n-1.$$

In a given triangulation $T \subset \{1, \ldots, m+n\}$, there are two possibilities for each edge e_i: either t_i and t_{i+1} are both at the top or both at the bottom, in which case either $i, i+1 \in T$ or $i, i+1 \notin T$; or else one triangle is at the top and the other at the bottom, in which case either $i \in T, i+1 \notin T$ or $i \notin T, i+1 \in T$. In the former case we say that e_i is of type "+" and colour the curve green in the toric diagram, in the latter case it is of type "−" and depict it in red. We let $\tau(e_i) = \pm 1$ according to whether e_i is of type "+" or "−".

4.2. **Computing triangulations.** Let us describe briefly how we verified the results with computer programs.

The actual triangulation was carried out using the software TOPCOM [TOP]. We used the function `points2allfinetriangs`, which triangulates a strip using triangles of equal, minimal area and produces a list of all possible triangulations.

Some details: In TOPCOM, points in a point set are given in homogeneous coordinates, so for our purposes, the vertex (i,j) corresponds to the point `[`i`,`j`,1]`. We label the $m+n+2$ vertices sequentially, assigning the range $0, \ldots, m$ to the vertices $v_0 :=$ `[0,0,1]`, $v_1 :=$ `[1,0,1]`, $\ldots$, $v_m :=$ `[`m`,0,1]`, and the range $m+1, \ldots, m+n+1$ to $v_{m+1} :=$ `[0,1,1]`, $v_{m+2} :=$ `[1,1,1]` $\ldots$, $v_{m+n+1} :=$ `[`n`,1,1]`. The output of TOPCOM consists of lists of triplets (v_a, v_b, v_c) of vertices giving the triangulation of the strip. Our task therefore was to extract the internal edges from this list and determine whether they are of type "+" or "−".

The natural ordering "from left to right" of the non-horizontal edges is precisely the the lexicographic ordering of either the top or the bottom vertices (i,j). When the edges are ordered in this fashion, the k^{th} edge, corresponding to vertex (i_k, j_k), is of type "+" if $j_{k-1} = j_k = j_{k+1}$ and $i_{k-1} + 1 = i_k = i_{k+1} - 1$; otherwise it is of type "−". (We are grateful to Jesus Martinez-Garcia for writing the program to compile this information.)

From this data we can quickly construct the partition function of any particular resolution of $C_{m,n}$ given by a specific triangulation.

5. Curve counting on singular varieties

For any complex threefold $(X, \mathcal{O}_X)$, the Hilbert scheme of ideal sheaves $\mathcal{I} \subset \mathcal{O}_X$ with fixed Euler characteristic $\chi(\mathcal{I}) = k$ and support $[\operatorname{supp}(\mathcal{I})] = \beta \in H_2(X;\mathbb{Z})$, written $I_k(X,\beta)$, has a perfect obstruction theory of virtual dimension $\int_\beta c_1(T_X) = -K_X(\beta)$, see [DT]. When $K_X = 0$, the numbers

$$N_{k,\beta}(X) := \int_{[I_k(X,\beta)]^{\text{vir}}} 1$$

are the *Donaldson–Thomas (DT) invariants* of X. We let $Q = (Q_1, \ldots, Q_h)$ be a set of symbols corresponding to generators of $H_2(X;\mathbb{Z})$. The DT invariants are collected into the *Donaldson–Thomas partition function*

$$Z(X; q, Q) := \sum_{k=0}^{\infty} \sum_{\beta \in H_2(X;\mathbb{Z})} N_{k,\beta}(X)\, q^k\, Q^\beta \ ,$$

where $Q^\beta = Q_1^{\beta_1} \cdots Q_h^{\beta_h}$. We single out the degree-0 contributions,

$$Z_0(X;q) := \sum_{k=0}^{\infty} N_{k,0}(X)\, q^k \ ,$$

and we define the *reduced DT partition function* as

$$Z'(X;q,Q) := Z(X;q,Q)/Z_0(X;q) \ .$$

The MacMahon function. In the sequel, we will require again and again the (generalised) MacMahon function

$$M(x,q) := \prod_{k=1}^{\infty} \frac{1}{(1-xq^k)^k} = \exp \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{i}{j} \ x^j \ q^{ij} \ .$$

For any smooth, toric, threefold X, we have $K_X(0) = 0$ and so we can define the degree-0 partition function $Z_0(X;q)$. It is known [MNOP1] that

$$Z_0(X;-q) = M(1,q)^{\int_X c_3(T_X \otimes K_X)} \ ,$$

and in particular if X is Calabi–Yau, then

$$Z_0(X;-q) = M(1,q)^{\chi(X)} \ ,$$

where $\chi(X)$ denotes the Euler characteristic of X. The connection is that the MacMahon function counts box partitions, and degree-0 toric ideal sheaves are given by monomial ideals, which can indeed be arranged like "boxes stacked into a corner".

5.1. **DT invariants of generalized conifolds.** If X is a crepant resolution of $C_{m,n}$, it is a smooth, toric, Calabi–Yau threefold, and the DT partition function can be computed combinatorially by the topological vertex method (see [LLLZ, IK]). We will always take the curves corresponding to the edges e_i as our preferred basis for $H_2(X;\mathbb{Z})$, that is,

$$\beta = \sum_{i=1}^{m+n-1} \beta_i [e_i] \in H_2(X;\mathbb{Z}) \ .$$

Furthermore, we have $\chi(X) = m+n$.

To describe $Z'(X;q,Q)$, we need to establish some notation. We call a set $P = \{i, i+1, \ldots, j\}$ an *edge path* if $1 \leq i \leq j \leq m+n-1$, and we think of it as a sequence of consecutive interior edges of the triangulation T of $C_{m,n}$ corresponding to the resolution X. An edge path P has *length* $|P| := j-i+1$. There are $m+n-1$ edge paths of length 1, $m+n-2$ of length 2, and so forth, and 1 of length $m+n-1$, so in total there are $\binom{m+n}{2}$ edge paths. An edge path is literally a path along the compact edges of the dual tropical curve of the triangulation T.

If $P = \{i, i+1, \ldots, j\}$ is an edge path, we write $Q_P = Q_{ij} = Q_i \cdots Q_j$, so for example $Q_{22} = Q_2$ and $Q_{35} = Q_3Q_4Q_5$. We define

$$f(P,q,Q) = M(Q_P,q)^{\tau(e_i)\tau(e_{i+1})\cdots\tau(e_j)} \ .$$

That is, $f(P,q,Q)$ is either the MacMahon function or its reciprocal, depending on whether P contains an even or an odd number of edges of type "$-$". The whole partition function of X is simply the product of such terms f over all edge paths:

$$Z'(X;-q,Q) = \prod_P f(P,q,Q) = \prod_{i=1}^{m+n-1} \prod_{j=i}^{m+n-1} \prod_{k=1}^{\infty} \Big(1 - \big(\textstyle\prod_{a=i}^{j} Q_a\big) \ q^k \Big)^{-k \prod_{a=i}^{j} \tau(e_a)} \ .$$

Since this partition function is determined entirely by the triangulation, i.e. by a subset $T \subset \{1,2,\ldots,m+n\}$, $|T| = m$, we write $Z'_T(C_{m,n};q,Q^T)$ for the partition function, where $Q^T = (Q^T_1, \ldots, Q^T_{m+n-1})$. We now consider the collection of all possible triangulations of $C_{m,n}$.

Definition 5.1. We define the *total partition function*:

$$Z'_{\mathrm{tot}}(C_{m,n};-q,Q) := \prod_{\substack{T \subset \{1,\ldots,m+n\} \\ |T|=m}} Z'_T(C_{m,n};-q,Q)$$

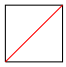
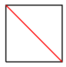
We can think of the total partition function as being a property of the singular variety itself. We make the following ad-hoc definition:

Definition 5.2. A partition function $Z(q,Q)$ of variables $Q=(Q_1,Q_2,\dots)$ is called *homogeneous* if

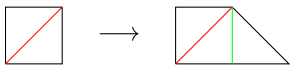
$$Z(q,Q)=\Big(\prod M\big(\textstyle\prod_{i\in A}Q_i,q\big)\Big)^d\ ,$$

where the first product is over an arbitrary finite collection of index sets $A\subset\{1,2,\dots\}$. The exponent d is called the *degree* of Z.

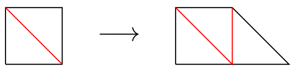
Example 5.3. We may start out with the case of $n=1$ and $m=1$, in which case the strip is just a single square admitting two triangulations:

 and 

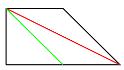
Thus the partition function reads $Z'_{\text{tot}}(C_{1,1};-q,Q)=M(Q_1,q)^{-2}$, for which the result is obviously true. Triangulations on smaller strips can be extended to triangulations of bigger strips. Consider the following two ways to pass from a triangulation of $C_{m,n}$ to a triangulation of $C_{m,n+1}$. In the first case, the right-most edge of $C_{m,n}$ turns into an internal edge of $C_{m,n+1}$ of "+" type, such as in this example:



The exponent of $M(Q_{1,m+n-1},q)$ coming from this triangulation of $C_{m,n}$ is the same as the exponent of $M(Q_{1,m+n},q)$ for the corresponding triangulation of $C_{m,n+1}$. Hence, there is a correspondence between such kinds of triangulations of the two strips, maintaining equality of exponents of the MacMahon factors. In the second case the rightmost edge of $C_{m,n}$ turns into an internal edge of $C_{m,n+1}$ of "−" type, such as in this figure:



Now notice that in the triangulation on the right-hand side we would have contributing factors of $M(Q_1,q)^{-1}$ and $M(Q_1Q_2,q)^{+1}$, which would appear to give rise to different exponents. However, since every parallelogram has 2 diagonals, there is always a second triangulation obtained by flopping the diagonal on rightmost parallelogram of the previous figure, and we obtain and extra triangulation of $C_{m,n+1}$ (this one not coming from a triangulation of $C_{m,n}$) which in the example in question gives:



and this triangulation contributes with factors of Q_1^{+1} and $(Q_1Q_2)^{-1}$, canceling out the seemingly unbalanced contributions from the previous one.

Proposition 5.4. *For $0<m\le n$, $Z'_{\text{tot}}(C_{m,n};-q,Q)$ is homogeneous of degree d, where*

$$d=\frac{(m^2-m+n^2-n-2mn)(m+n-2)!}{m!n!}\ , \tag{5.1}$$

namely,

$$Z'_{\text{tot}}(C_{m,n};-q,Q)=\prod_{1\le i\le j\le m+n-1}M(Q_{ij},q)^d\ ,$$

Proof. The proposition consists of two separate parts, and so does the proof. The first statement is that each MacMahon factor $M(Q_{ij},q)$ appears with the *same* power in the total partition function.

We have to show that each MacMahon factor $M(Q_{ij},q)$ appears with the *same* power in the total partition function and compute the value of this exponent. The problem is entirely combinatorial. In terms of finite sets, it takes the following form: Let us simply write N for the finite set $\{1,2,\dots,N\}$. For any subset $T\subset N$ and any fixed, ordered subset $S=\{s_1,\dots,s_k\}\subset N$, we define the *characteristic sequence*

$$\chi_T(S):=\big(\chi_T(s_1),\dots,\chi_T(s_k)\big)\ ,$$

where $\chi_T\colon N\to\{0,1\}=\mathbb{Z}/2\mathbb{Z}$ is the characteristic function of T. (It will be opportune to think of the two-element set as the additive group of order 2.)

In our application, we will take S to be a "contiguous" subset of the form $\{i, i+1, \dots, j\}$ corresponding to some edge path. For such a subset, we define the *difference sequence* as

$$\Delta_T(S) := \big(\chi_T(s_1) - \chi_T(s_2), \chi_T(s_2) - \chi_T(s_3), \dots, \chi_T(s_{k-1}) - \chi_T(s_k)\big) ,$$

and we define the T*-signature* of S as

$$\sigma_T(S) := \prod_{b \in \Delta_T(S)} (-1)^b \in \{+1, -1\} .$$

(Since we are only interested in the T-signature, we may consider the elements of $\Delta_T(S)$ to take values in $\mathbb{Z}/2\mathbb{Z}$ and identify $+1$ and -1.) Finally, the exponent of $M(Q_{ij}, q)$ in the total partition function of $C_{m,n}$ is the m*-signature* of the set $S = \{i, i+1, \dots, j\}$, defined as

$$\sigma(S) = \sum_{T \subset N \,:\, |T| = m} \sigma_T(S) ,$$

where $N = m + n$.

So much for the setup. The first observation is that any action $\pi \in \Sigma_N$ that preserves the contiguous ordering of the elements of S does not alter the value of the total signature: $\sigma(\pi S) = \sigma(S)$. Therefore, we may assume without loss of generality that S is $\{1, 2, \dots, k\}$.

Next, any subset $T \subset N$ with $|T| = m$ is of the form $T = U \sqcup T'$, where $U \subset \{1, 2, \dots, k\}$ with $|U| = i$ and $T' \subset \{k+1, k+2, \dots, N\}$ with $|T'| = m - i$ for $i = 0, 1, \dots, k$. Now observe that all we need to compute the m-signature is $\Delta_U(S)$, or rather $\sigma_U(S) = \sigma_T(S)$. Since there are $\binom{N}{m}$ subsets in total, we have

$$\sigma(S) = \big|\{T : \sigma_T(S) = +1\}\big| - \big|\{T : \sigma_T(S) = -1\}\big| = \binom{N}{m} - 2\,\big|\{T : \sigma_T(S) = -1\}\big| .$$

The combinatorics of this are easily determined: Subsets $T = U \sqcup T'$ for which $\sigma_U(S) = -1$ are those for which $\Delta_U(S)$ has an odd number of 1s, and there are $2\binom{k-2}{i-1}$ of those, where $i = |U|$. Summing over all i we find:

$$\sigma(S) = \binom{N}{m} - 4 \sum_{i=1}^{k-1} \binom{k-2}{i-1} \binom{N-k}{m-i} .$$

The last factor accounts for all the possible subsets T'. The sum evaluates to $\binom{N-2}{m-1}$, and we obtain:

$$\sigma(S) = \binom{N}{m} - 4\binom{N-2}{m-1} = \frac{(N^2 - N + 4m^2 - 4mN)(N-2)!}{m!(N-m)!}$$

This is true for any contiguous set $S = \{i, i+1, \dots, j\}$, and the result follows by substituting $N = m + n$.

The second statement is the value of the exponent. Since the exponent is the same for each factor $M(Q_{ij}, q)$ by the first part, we may compute it by just computing the exponent of $M(Q_1, q)$, i.e. the factor corresponding to the edge path $\{1\}$. Each triangulation T contributes either an exponent $+1$ or -1. The exponent is $+1$ if $1, 2 \in T$ or $1, 2 \notin T$, and it is -1 if $1 \in T$, $2 \notin T$ or if $1 \notin T$, $2 \in T$. The number of $+1$s is thus the sum of the number of triangulations of $C_{m-2,n}$ and $C_{m,n-2}$, and the number of -1s is twice the number of triangulations of $C_{m-1,n-1}$. □

Remark 5.5. We excluded the case $n > m = 0$ from the proposition, since $C_{0,n}$ only admits one unique triangulation, and all interior edges are of type "+". Writing X for the resolution, we have

$$Z'(X; -q, Q) = \prod_{1 \le i \le j \le n-1} M(Q_{ij}, q) \qquad \text{and} \qquad Z(X; -q, Q) = M(1, q)^n \, Z'(X; -q, Q) .$$

We have indeed $d = 1$ in Equation 5.1 whenever $m = 0$.

Results. All toric Calabi–Yau threefolds are non-compact and described completely by their toric fan, whose ray generators all lie in the $\{z = 1\}$-plane, and the intersection of the toric fan with this plane is a compact polytope. The condition that the threefold contains no compact 4-cycles is equivalent to the condition that the polytope contains no interior lattice points. The only polytopes that satisfy this condition are the strips $C_{m,n}$, so the only toric Calabi–Yau threefolds without compact 4-cycles are either quotients of $\mathbb{C}^3$ by $\mathbb{Z}/n\mathbb{Z}$, whose polytope is $\mathbb{C}_{0,n}$, or quotients of the conifold $\{xy - zw = 0\}$ by $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, whose polytope is $C_{m,n}$; or one of their resolutions.

Definition 5.6. Let us call a partition function defined for a Calabi–Yau manifold Y to be of *curve-counting type* if it can be expressed in terms of the Donaldson–Thomas partition function up to some factor depending only on the Euler characteristic of Y.

We have thus proved:

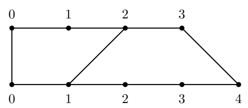
Theorem 5.7. *Let X be a toric Calabi–Yau threefold without compact 4-cycles and without contractible curves, and let $Z(Y;q,Q)$ be any partition function of curve-counting type. Then the total partition function*

$$Z_{\text{tot}}(X;q,Q) := \prod_{Y\to X} Z(Y;q,Q) \ ,$$

where the product ranges over all crepant resolutions of X, is homogeneous, and its degree is given by Proposition 5.4.

Note that the absence of compact 4-cycles implies that there are no interior points in the planar polytope that represents the toric CY threefold, from which it follows that this polytope is a strip, hence the threefold is one of our generalised conifolds.

For a general toric Calabi–Yau threefold X without compact 4-cycle, we can use this theorem to factor the partition function into homogeneous factors. The toric diagram Δ of X is a strip of shape $C_{m,n}$ with an arbitrary number of internal edges filled in, for example:



Let us partition the integers m,n according to the already filled-in interior edges, that is,

$$(m,n) = \sum_{k=1}^{P}(m_k,n_k) = (m_1+m_2+\cdots+m_P, n_1+n_2+\cdots+n_P) \ .$$

In the example above, we have $(m,n)=(4,3)$, and the single interior edge corresponds to the partition $(4,3)=(1+3,2+1)$. It is clear that the number of maximal triangulations of this shape is

$$\prod_{k=1}^{P}\binom{m_k+n_k}{n_k} \ ,$$

where each factor counts the number of triangulations of the embedded subdiagram $C_{m_k,n_k} =: C_k$. If we restrict our attention to some fixed subdiagram C_k, then entire collection of triangulations of Δ contains many triangulations with the same restriction to C_k. It is clear that for any fixed triangulation of C_k, there are b_k triangulations of Δ that restrict to the given triangulation, where

$$b_k = \prod_{j\neq k}\binom{m_j+n_j}{n_j} \ .$$

We extend Definition 5.1 the straightforward way:

Definition 5.8. If X is a Calabi–Yau threefold without compact 4-cycles such that the convex hull of its toric diagram is $C_{m,n}$ (that is, there exists a birational map $X\to C_{m,n}$), we define the *total partition function* to be

$$Z'_{\text{tot}}(X;-q,Q) := \prod_{T} Z'_T(C_{m,n},-q,Q) \ .$$

Here the term in the product of the right-hand side is the same as in Definition 5.1, but time the product is taken only over those triangulations T which correspond to resolutions of X.

Now Theorem 5.7 implies the following:

Corollary 5.9. *If X is a toric Calabi–Yau threefold without compact 4-cycles and (m,n), P and b_k are as above, then the total partition function of X factors as follows:*

$$Z'_{\text{tot}}(X;-q,Q) = Z''(-q,Q)\prod_{k=1}^{P} Z'_{\text{tot}}(C_{m_k,n_k};-q,Q)^{b_k} \ .$$

The factors in the product on the right are homogeneous as per Theorem 5.7, and the function Z'' only contains factors $M(Q_{ij}, q)$ for which the edge path corresponding to Q_{ij} crosses one of the interior edges of the toric diagram of X.

Example 5.10. In the above example with $(m, n) = (4, 3) = (1 + 3, 2 + 1)$, the two homogeneous factors are $Z'_{\mathrm{tot}}(C_{1,2}; -q, Q)^3$ and $Z'_{\mathrm{tot}}(C_{3,1}; -q, Q)^2$, and the inhomogeneous factor contains only terms $M(Q_{ij}, q)$ with $i \leq 3 \leq j$, because the third edge is already fixed in the diagram.

5.2. **BPS counting and relation to black holes.** Here is one application to BPS state counting. The *topological string partition function* of X is

$$Z_{\mathrm{top}}(X; q, Q) = M(1, q)^{\chi(X)/2} Z'(X; -q, Q) ,$$

so it is a partition function of curve-counting type.

Corollary 5.11. *Writing X_T for the resolution of $C_{m,n}$ corresponding to the triangulation T, we have*

$$\prod_T Z_{top}(X_T; q, Q) = M(1, q)^{\binom{m+n}{m}\frac{m+n}{2}} \prod_{1 \leq i \leq j \leq m+n-1} M(Q_{ij}, q)^{\frac{(m^2-m+n^2-n-2mn)(m+n-2)!}{m!n!}} .$$

Proof. This follows immediately from the fact that $\chi(X_T) = m + n$ for all T and that there are $\binom{m+n}{m}$ triangulations. □

We finish up with a comment on the relation between BPS counting and black holes. The counting of BPS states is of great interest to string theory and supergravity, and it has been shown in several situations (see e.g. [S1, IS]) that the counting of BPS states agrees with the counting of extremal black holes. In some cases it has even been shown [S2] that the string partition function agrees with the black hole partition function; Sen concludes from this agreement a precise equivalence between the black hole entropy and the statistical entropy associated with an ensemble of BPS states.

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