LG MODELS AS SYMPLECTIC LEFSCHETZ FIBRATIONS ON ADJOINT ORBITS

ELIZABETH GASPARIM, LINO GRAMA, AND LUIZ A. B. SAN MARTIN

ABSTRACT. We prove that adjoint orbits of semisimple Lie algebras have the structure of symplectic Lefschetz fibrations. These provide a large class of examples of LG models whose superpotential has only nondegenerate critical points. We describe the topology of the regular and singular fibres, in particular calculating their middle Betti numbers.

CONTENTS

1. INTRODUCTION

A Landau–Ginzburg model (LG) is a nonlinear sigma model on a space X together with a superpotential W. The superpotential $W\colon X \to \mathbb{C}$ is required to be holomorphic, so LG models are only interesting when X is noncompact. See for instance [GSh] or [KP] for some examples of LG models with nontrivial X. Here we contribute with a large class of LG models where X is an adjoint orbit of a semisimple Lie algebra, and the superpotential provides X with the structure of a Symplectic Lefschetz Fibrations (SLF). These are particularly manageable examples of LG models because for an SLF the superpotential has only of nondegenerate critical points.

1.1. **Our first motivation: Homological Mirror Symmetry.** Given any complex variety the celebrated Homological Mirror Symmetry conjecture of Kontsevich [Ko] predicts the existence of a symplectic mirror partner such that the category of A-branes (Lagrangian thimbles) $D(Lag(W))$ is equivalent to the derived category of B-branes (coherent sheaves) $D^b(Coh(Y))$. Here $D(Lag(W))$ is the Fukaya–Seidel category of vanishing cycles for a LG model $W\colon X \to \mathbb{C}$ and $D^b(Coh(Y))$ is the bounded derived category of coherent sheaves on Y. An exciting part of the conjecture is that the A-side is symplectic geometry whereas the B-side is algebraic, therefore the conjecture provides a dictionary between the two types of geometry – algebraic and symplectic – the mirror

The authors acknowledge support of Fapesp grant 2012/10179-5, Fapesp grant 2012/07482-8 , Fapesp grant 2012/18780-0 and CNPq grant 303755/2009-1.

map interchanging vanishing cycles on the symplectic side with coherent sheaves on the algebraic side.

The HMS conjecture generated an enormous amount of interest in both the Physics and the Mathematics community. Mathematical proofs appeared in several cases: elliptic curves by Polishschuk–Zaslow [PZ], curves of genus two by Seidel [Se], curves of higher genus by Efimov [E], punctured spheres by Abouzaid–Auroux–Efimov–Katzarkov–Orlov [AAEKO], weighted projective planes and del-Pezzo surfaces by Auroux–Katzarkov–Orlov [AKO1], [AKO2], quadrics and intersection of two quadrics by Smith [S], the four torus by Abouzaid–Smith [AbS], Calabi–Yau hypersurfaces in projective space by Sheridan [Sh], toric varieties by Abouzaid [Ab], and Abelian varieties by Fukaya [F]. Nevertheless, the HMS conjecture remains open in most cases.

Clarke [Cl] showed that one can state a generalized version of the HMS conjecture as a duality between LG models. He also shows that this correspondence generalizes those of Batyrev–Borisov Berglung–Hübsch, Givental, and Hori–Vafa. Thus, LG models are basic tools to the study of the HMS conjecture.

1.2. **Our second motivation: existence of SLFs in higher dimensions.** The literature about SLFs in 4 real dimensions is vast. In fact, in 4D a celebrated result of Donaldson [Do] proves that after blowing up finitely many points, every symplectic manifold admits a Lefschetz fibration. On the opposite direction, the existence of a topological Lefschetz fibration on a 4 dimensional symplectic manifold guaranties the existence of an SLF whenever the fibres have genus at least 2, see [GoS]. Moreover, Amorós–Bogomolov–Katzarkov–Pantev proved existence SLFs in 4D with arbitrary fundamental group [ABKP].

In general, it is possible to construct Lefschetz fibrations starting up with a Lefschetz pencil and then blowing up its base locus (see [Se], [Go]). However, in such cases one needs to fix the indefiniteness of the symplectic form over the exceptional locus by glueing in a correction, and this makes it rather difficult to explicitly find vanishing cycles and thimbles. Direct constructions of Lefschetz fibrations in higher dimensions are by and large lacking in the literature.

Our goal here is to investigate the existence of SLFs on complex n-folds with $n \geq 3$. Our construction does not make use of Lefschetz pencils, we construct our symplectic Lefschetz fibrations directly taking the superpotentials provided by heigh functions that comes naturally from Lie theory.

1.3. **Main results.** We prove that adjoint orbits of semisimple Lie algebras have the structure of symplectic Lefschetz fibrations. We then describe the topology of the fibres, in particular calculating their middle Betti numbers. Our main results are:

Theorem 2.2 Let $\mathfrak{h}$ be the Cartan subalgebra of a complex semisimple Lie algebra. Given $H_0 \in \mathfrak{h}$ and $H \in \mathfrak{h}_{\mathbb{R}}$ with H a regular element. The *height function* $f_H : \mathscr{O}(H_0) \to \mathbb{C}$ defined by

$$f_H(x) = \langle H, x \rangle \qquad x \in \mathscr{O}(H_0)$$

has a finite number ($= |\mathscr{W}|/|\mathscr{W}_{H_0}|$) of isolated singularities and gives $\mathscr{O}(H_0)$ the structure of a symplectic Lefschetz fibration.

The precise meaning of this statement is explained in section 2.1, and comments about our choice of f_H are given in remark 2.3. In example 2.5 we describe the category of Lagrangian vanishing cycles for an adjoint orbit of the lie algebra $\mathfrak{sl}(2,\mathbb{C})$. In section 3 we describe the topology of the regular fibre, and in section 4 we describe the singular fibre, obtaining:

Corollary 3.5 The homology of a regular level $L(\xi)$ coincides with that of $\mathbb{F}_{H_0} \setminus \mathscr{W} \cdot H_0$. In particular, the middle Betti number of $L(\xi)$ equals $k-1$, where k is the number of singularities of the fibration f_H (and equals the number of elements in the orbit $\mathscr{W} \cdot H_0$).

Corollary 4.2 The homology of a singular level $L(wH_0)$, $w \in \mathscr{W}$ coincides with that of

$$\mathbb{F}_{H_0} \setminus \{uH_0 \in \mathscr{W} \cdot H_0 : u \neq w\}.$$

In particular, the middle Betti number of $L(wH_0)$ equals $k-2$, where k is the number of singularities of the fibration f_H.

Acknowledgments We thank Ron Donagi, Ludmil Katzarkov and Tony Pantev for enlightening discussions.

2. Lefschetz Fibrations on adjoint orbits

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and G a connected Lie group with Lie algebra $\mathfrak{g}$ (for instance G could be $\mathrm{Aut}_0(\mathfrak{g})$, the connected component of the identity of the automorphism group of G).

The Cartan–Killing form of $\mathfrak{g}$, $\langle X, Y\rangle = \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)) \in \mathbb{C}$, is symmetric and nondegenerate. Moreover, $\langle\cdot,\cdot\rangle$ is invariant by the adjoint representation, that is

$$\langle [X,Y], Z\rangle = -\langle Y, [X,Z]\rangle \qquad X, Y, Z \in \mathfrak{g}.$$

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a compact real form $\mathfrak{u}$ of $\mathfrak{g}$. Associated to these subalgebras there are the subgroups $T = \langle \exp \mathfrak{h}\rangle = \exp \mathfrak{h}$ and $U = \langle \exp \mathfrak{u}\rangle = \exp \mathfrak{u}$. Denote by τ the conjugation associated to $\mathfrak{u}$, defined by $\tau(X) = X$ if $X \in \mathfrak{u}$ and $\tau(Y) = -Y$ if $Y \in i\mathfrak{u}$. Hence if $Z = X + iY \in \mathfrak{g}$ with $X, Y \in \mathfrak{u}$ then $\tau(X+iY) = X - iY$. In this case, the sesquilinear form $\mathscr{H}_\tau : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by

$$\mathscr{H}_\tau(X,Y) = -\langle X, \tau Y\rangle \tag{2.1}$$

is a Hermitian form on $\mathfrak{g}$ (see [SM, lemma 12.17]).

A root of $\mathfrak{h}$ is a linear functional $\alpha : \mathfrak{h} \to \mathbb{C}$, $\alpha \neq 0$, such that the space of roots

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \forall H \in \mathfrak{h},\ [H,X] = \alpha(H)\,X\} \neq \{0\}.$$

The set of all roots is denoted by Π. The decomposition $\mathfrak{g}$ in eigenspaces of $\mathrm{ad}(H)$, $H \in \mathfrak{h}$, is given by

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha.$$

An element $H \in \mathfrak{h}$ is **regular** if $\alpha(H) \neq 0$ for all $\alpha \in \Pi$.

The restriction of the Cartan–Killing form to $\mathfrak{h}$ is nondegenerate so we can define, for each $\alpha \in \Pi$, $H_\alpha \in \mathfrak{h}$ by $\alpha(\cdot) = \langle H_\alpha, \cdot\rangle$. The real subspace generated by H_α, $\alpha \in \Pi$, is denoted by $\mathfrak{h}_\mathbb{R}$. In the canonical construction of $\mathfrak{u}$ we have $\mathfrak{h}_\mathbb{R} \subset i\mathfrak{u}$.

The Weyl group $\mathscr{W}$ is given by $\mathscr{W} = \mathrm{Nor}_G(\mathfrak{h})/\mathrm{Cent}_G(\mathfrak{h})$ (normaliser modulo centraliser) or, equivalently, the group generated by reflexions with respect to the roots. $\mathscr{W}$ is finite.

The adjoint representation of G in $\mathfrak{g}$ is denoted by $\mathrm{Ad}(g)\,X$, $g \in G$ and $X \in \mathfrak{g}$, or simply by $g \cdot X$. An adjoint orbit is given by

$$\mathscr{O}(X) = G \cdot X = \{g \cdot X \in \mathfrak{g} : g \in G\}.$$

Such an orbit can be identified with the quotient space $G/\mathrm{Cent}_G(X)$ where $\mathrm{Cent}_G(X) = \{g \in G : g \cdot X = X\}$ is the *centraliser* of X in G. If $H \in \mathfrak{h}$ is regular then $\mathrm{Cent}_G(H) = T = \exp \mathfrak{h}$. The tangent space $T_x\mathscr{O}(X)$ to the orbit $\mathscr{O}(X)$ at x is given by

$$\begin{aligned} T_x\mathscr{O}(X) &= \Im \mathrm{ad}(x) = \{[x,A] : A \in \mathfrak{g}\} \\ &= \{[A,x] : A \in \mathfrak{g}\} \end{aligned}$$

since $[A,x] = \frac{d}{dt}\left(e^{t\mathrm{ad}(A)}x\right)_{|t=0}$ and $e^{t\mathrm{ad}(A)} = \mathrm{Ad}\left(e^{tA}\right)$.

Note that, because $\mathfrak{g}$ is a complex Lie algebra, the tangent spaces $T_x\mathscr{O}(X)$ to $\mathscr{O}(X)$ are complex subspaces of $\mathfrak{g}$, since if $[A,x]$ is a tangent vector then $i[A,x] = [iA,x]$ is also a tangent vector. This implies that each adjoint orbit $\mathscr{O}(X)$ is a complex manifold, as it is endowed with an almost complex structure (multiplication by i in each tangent space) which is integrable, simply because this almost complex structure is the restriction of a complex structure on $\mathfrak{g}$ (the Nijenhuis tensor vanishes).

Example 2.1. When $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ the data just described is:

(1) $\langle\cdot,\cdot\rangle$ is a (constant) multiple of the form $\mathrm{tr}(XY)$;
(2) A canonical choice of $\mathfrak{h}$ is the subalgebra of diagonal matrices;
(3) with this choice of $\mathfrak{h}$ the roots are the linear functionals $\alpha_{ij}\left(\mathrm{diag}\{a_1,\dots,a_n\}\right) = a_i - a_j$, $i \neq j$, with $\mathfrak{g}_{\alpha_{ij}}$ the subspace generated by the basis element given by the matrix E_{ij} (with 1 in the i,j entry and zeros elsewhere);
(4) $\mathfrak{u} = \mathfrak{su}(n)$, the (real) algebra of anti-Hermitian matrices. In this case $\tau(Z) = -\overline{Z}^T$, $Z \in \mathfrak{sl}(n,\mathbb{C})$ and the associated Hermitian form is a multiple of $\mathscr{H}_\tau(X,Y) = \mathrm{tr}\left(X\overline{Y}^T\right)$;
(5) $H \in \mathfrak{h}$ is regular if and only if its eigenvalues are all distinct;
(6) $\mathscr{W}$ is the permutation group of n elements, which acts upon $\mathfrak{h}$ by permuting its diagonal entries.
(7) If $H \in \mathfrak{h}$ then $\mathscr{O}(H)$ is the set of diagonalizable matrices that have the same eigenvalues as H.

2.1. Main Theorem.

The Lefschetz fibration on an adjoint orbit is the following:

Theorem 2.2. *Given $H_0 \in \mathfrak{h}$ and $H \in \mathfrak{h}_{\mathbb{R}}$ with H a regular element. Then, the "height function" $f_H : \mathscr{O}(H_0) \to \mathbb{C}$ defined by*

$$f_H(x) = \langle H, x\rangle \qquad x \in \mathscr{O}(H_0)$$

has a finite number ($= |\mathscr{W}|/|\mathscr{W}_{H_0}|$) of isolated singularities and defines a symplectic Lefschetz fibration, that is, the following properties hold:

(1) *The singularities are nondegenerate (Hessian non degenerate).*
(2) *If $c_1, c_2 \in \mathbb{C}$ are regular values then the level manifolds $f_H^{-1}(c_1)$ and $f_H^{-1}(c_2)$ are diffeomorphic.*
(3) *There exists a symplectic form Ω in $\mathscr{O}(H_0)$ such that if $c \in \mathbb{C}$ is a regular value then the level manifold $f_H^{-1}(c)$ is symplectic, that is, the restriction of Ω to $f_H^{-1}(c)$ is a symplectic (nondegenerate) form.*
(4) *If $c \in \mathbb{C}$ is a singular value, then $f_H^{-1}(c)$ contains affine subspaces (contained in $\mathscr{O}(H_0)$). These subspaces are symplectic with respect to the form Ω from the previous item.*

The proof will be carried out in several steps.

Remark 2.3. The height function f_H defined by an element $H \in \mathfrak{h}_{\mathbb{R}}$ is extensively used in the study of the geometry of flag manifolds. This is due to the fact that it is a Morse–Bott function in general, which is Morse if H is regular. These height functions make the link between Morse theory and the algebraic theory of Bruhat decompositions. This is because the gradient $\mathrm{grad}\, f_H$ of f_H, with respect to the so called Borel metric is precisely the vector field $\widetilde{H}$ induced by H on a flag manifold (see Duistermaat–Kolk–Varadarajan [DKV]). The unstable manifolds of $\mathrm{grad}\, f_H = \widetilde{H}$ are the components of the Bruhat decomposition if H is regular. For applications of these height functions to the geometry of flag manifolds see Kocherlakota [Kc], regarding the Morse homology, and the extensive literature on the "convexity theorems" started with Kostant [K], Atiyah [At] and Guillemin–Sternberg [GS].

2.2. **Singular points of the potential as an orbit of the Weyl group.** First of all, if $A \in \mathfrak{g}$ and $x \in \mathcal{O}(H_0)$ then $[A, x]$ is a vector tangent to $\mathcal{O}(H_0)$ at x and the differential of f_H is given by

$$(df_H)_x([A,x]) = \frac{d}{dt}\langle H, e^{t\operatorname{ad}(A)}x\rangle_{|t=0} = \langle H, [A,x]\rangle = \langle [x,H], A\rangle. \tag{2.2}$$

From this expression it follows that f_H is a holomorphic function with respect to the complex structure of $\mathcal{O}(H_0)$. Indeed,

$$(df_H)_x(i[A,x]) = (df_H)_x([iA,x]) = \langle [x,H], iA\rangle = i\langle [x,H], A\rangle = i\,(df_H)_x([A,x]).$$

Being a holomorphic function, the rank of f_H at $x \in \mathcal{O}(H_0)$ (regarded as a map taking values in $\mathbb{R}^2 \approx \mathbb{C}$) is either 0 or 2, given that if $(df_H)_x([A,x]) \neq 0$ then $i\,(df_H)_x([A,x]) \neq 0$ and these two derivatives generate $\mathbb{R}^2 \approx \mathbb{C}$. In particular, this means that $x \in \mathcal{O}(H_0)$ is a singular point of f_H if and only if $(df_H)_x = 0$.

Therefore, by expression (2.2) for the differential of f_H, it follows that x is a singularity, that is, $(df_H)_x([A,x]) = 0$ for all $A \in \mathfrak{g}$ if and only if $[x, H] = 0$. This allows us to identify the singular points.

Proposition 2.4. *x is a singular point for f_H if and only if $x \in \mathcal{O}(H_0) \cap \mathfrak{h} = \mathcal{W} \cdot H_0$, where $\mathcal{W}$ is the Weyl group. (At this point the hypothesis that H is regular is used.)*

Proof. As observed, x is a singularity if and only if $[x, H] = 0$. But, as H is regular its centralizer is the Cartan subalgebra $\mathfrak{h}$ itself. It follows that the singularity set is $\mathcal{O}(H_0) \cap \mathfrak{h}$. This set is exactly the orbit of H_0 by the action of $\mathcal{W}$. □

Since $\mathcal{W}$ is finite we obtain the following corollary.

Corollary 2.5. *The set of singularities of f_H is finite.*

To obtain the Hessian at a singularity $x_0 \in \mathcal{O}(H_0) \cap \mathfrak{h}$, take $B \in \mathfrak{g}$. Then the second derivative at $x \in \mathcal{O}(H_0)$ calculated at $[A, x]$ and $[B, x]$ is given by

$$\begin{aligned} \frac{d}{dt}\langle [e^{t\operatorname{ad}(B)}x, H], A\rangle_{|t=0} &= \langle [[B,x],H], A\rangle \\ &= \langle [[B,H],x], A\rangle + \langle [B,[x,H]], A\rangle. \end{aligned}$$

In particular, if x_0 is a singularity then $[x_0, H] = 0$ and the second derivative becomes

$$\langle [[B,H],x_0], A\rangle = \langle [x_0,[H,B]], A\rangle. \tag{2.3}$$

Proposition 2.6. *The second term of (2.3) defines a symmetric bilinear form whose restriction to the tangent space $T_{x_0}\mathcal{O}(H_0)$ at $x_0 \in \mathfrak{h}$ is nondegenerate.*

Proof. The tangent space $T_{x_0}\mathcal{O}(H_0)$ is the image of $\operatorname{ad}(x_0)$, which equals

$$\operatorname{im}(\operatorname{ad}(x_0)) = \sum_{\alpha(x_0)\neq 0} \mathfrak{g}_\alpha$$

given that $\operatorname{ad}(x_0)$ is diagonalizable and its eigenvalues are 0 and $\alpha(x_0)$, $\alpha \in \Pi$. From this we observe that the restriction of $\operatorname{ad}(x_0)$ to its image is an invertible linear map. Therefore, the tangent vectors $[x_0, A]$ with A varying inside $\operatorname{im}(\operatorname{ad}(x_0))$ cover the entire tangent space $T_{x_0}\mathcal{O}(H_0)$. This means that in the second derivative (2.3) we can restrict A and B to $\operatorname{im}(\operatorname{ad}(x_0))$.

Now, on one hand the restriction of $\operatorname{ad}(H)$ to $\operatorname{im}(\operatorname{ad}(x_0))$ is also invertible since H is regular. On the other hand, the restriction of the Cartan–Killing form to $\operatorname{im}(\operatorname{ad}(x_0))$ is nondegenerate, since if $\alpha(x_0) \neq 0$ then $(-\alpha)(x_0) \neq 0$ and given $Y \in \mathfrak{g}_\alpha$ there exists $Z \in \mathfrak{g}_{-\alpha}$ such that $\langle Y, Z\rangle \neq 0$.

The upshot is that the expression $\langle [x_0,[H,B]], A\rangle$ with $A, B \in \operatorname{im}(\operatorname{ad}(x_0))$ takes the form $\mathfrak{B}(Pu, v)$ where $\mathfrak{B}$ is a nondegenerate bilinear form and P is an invertible linear transformation on a vector space. Such a bilinear form is always nondegenerate. □

This proposition concludes the proof of item (1) of theorem 2.2.

2.3. **Diffeomorphisms among regular fibres.** To show that the inverse images of two regular points are diffeomorphic, we construct vector fields transversal to the fibres in such a way that for a given fibre the flows of these vectors fields are well defined up to a certain time in all the fibre (as $\mathscr{O}(H_0)$ is not compact, it is not to be expected that the vector fields be complete). The diffeomorphism is obtained form such flows.

The transversal vector fields that will play the appropriate roles are defined by

$$Z(x) = \frac{1}{\|[x,H]\|^2}[x,[\tau x,H]] \tag{2.4}$$

where $\tau : \mathfrak{g} \to \mathfrak{g}$ is conjugation with respect to the real compact form $\mathfrak{u}$ and $\|\cdot\|$ is the norm associated to the Hermitian form $\mathscr{H}$. Here are a few observations about this vector field:

(1) Z is well defined if $[x,H] \neq 0$, that is, if $x \notin \mathfrak{h}$. Therefore, Z can be regarded as a vector field on $\mathfrak{g} \setminus \mathfrak{h}$, which restricts to a vector field on the set of regular points of $\mathscr{O}(H_0) \setminus \mathfrak{h}$.
(2) If $x \in \mathscr{O}(H_0) \setminus \mathfrak{h}$ then $Z(x)$ is tangent to $\mathscr{O}(H_0)$ since $[x,[\tau x,H]] \in \operatorname{im}(\operatorname{ad}(x))$ is tangent to $\mathscr{O}(H_0)$ at x. Therefore, Z does indeed restrict to a vector field in $\mathscr{O}(H_0) \setminus \mathfrak{h}$.
(3) Since, by hypothesis, for $H \in \mathfrak{h}_{\mathbb{R}}$, $\tau H = -H$ it follows that $[\tau x, H] = -[\tau x, \tau H] = -\tau[x,H]$.
(4) The differential of f_H at $x \in \mathscr{O}(H_0) \setminus \mathfrak{h}$ satisfies
$$\begin{aligned}(df_H)_x([x,[\tau x,H]]) &= -\langle H,[x,[\tau x,H]]\rangle = \langle H,[x,\tau[x,H]]\rangle \\ &= -\langle [x,H],\tau[x,H]\rangle \\ &= \mathscr{H}([x,H],[x,H]) = \|[x,H]\|^2\end{aligned}$$
which is > 0 if $[x,H] \neq 0$. Therefore, $df_H(Z(x)) = 1$. This guarantees that Z is transversal to the level surfaces of f_H.
(5) The vector field iZ is also transversal. This happens because the tangent spaces to a level surface $f_H^{-1}(c)$, for a regular value $c \in \mathbb{C}$, are complex subspaces of $\mathfrak{g}$. Therefore if $Z(x) \notin T_x f_H^{-1}(c)$ then $iZ(x) \notin T_x f_H^{-1}(c)$.

Lemma 2.7. *Let $Z : \mathfrak{g} \setminus \mathfrak{h} \to \mathfrak{g}$ be defined by*

$$Z(x) = \frac{1}{\|[x,H]\|^2}[x,[\tau x,H]]$$

where $\|\cdot\|$ is the norm corresponding to the Hermitian form $\mathscr{H}(\cdot,\cdot)$. Then, there exists $M > 0$ such that for all $x \in \mathfrak{g} \setminus \mathfrak{h}$ the following inequality holds

$$\|dZ_x\| \leq 2M(\|\operatorname{ad}(H)\| + M\|H\|)\frac{\|x\|}{\|[x,H]\|^2}.$$

The constant $M > 0$ depends only on the bracket of $\mathfrak{g}$.

Proof. It suffices to show that the differential of Z, dZ_x is bounded as a function of x. If $v \in \mathfrak{g}$ then

$$dZ_x(v) = -\frac{2\Re\mathscr{H}([v,H],[x,H])}{\|[x,H]\|^4}[x,[\tau x,H]] + \frac{1}{\|[x,H]\|^2}([v,[\tau x,H]] + [x,[\tau v,H]]).$$

To estimate $\|dZ_x(v)\|$ (and thus also $\|dZ_x\|$) we use the following inequalities:

(1) $|\Re\mathscr{H}([v,H],[x,H])| \leq |\mathscr{H}([v,H],[x,H])| \leq \|[x,H]\|\cdot\|\operatorname{ad}(H)\|\cdot\|v\|$, by the Cauchy–Schwarz inequality, where $\|\operatorname{ad}(H)\|$ is the operator norm of $\operatorname{ad}(H)$.
(2) The bracket of a finite dimensional Lie algebra is a continuous bilinear map, hence there exists $M > 0$ such that for all $X, Y \in \mathfrak{g}$ we have $\|[X,Y]\| \leq M\|X\| \cdot \|Y\|$. Consequently,

(a) $\|[x,[\tau x,H]]\| \le M\|[\tau x,H]\|\cdot\|x\|$. Since τ is an isometry of the Hermitian form $\mathscr{H}$ and $H\in\mathfrak{h}_{\mathbb{R}}$, $\|[\tau x,H]\| = \|-\tau[x,H]\| = \|[x,H]\|$. Therefore, the second term of this inequality equals $M\|[x,H]\|\cdot\|x\|$.
(b) $\|[v,[\tau x,H]]\|$ e $\|[x,[\tau v,H]]\|$ are bounded above by $M^2\|H\|\cdot\|x\|\cdot\|v\|$.

An application of the triangle inequality to $\|dZ_x(v)\|$, combined with the previous expression, gives us

$$\|dZ_x(v)\| \le 2\left(\frac{M\|\mathrm{ad}(H)\|\cdot\|x\|}{\|[x,H]\|^2} + \frac{M^2\|H\|\cdot\|x\|}{\|[x,H]\|^2}\right)\|v\|,$$

from which the claimed inequality follows. □

Now we find estimates for $\frac{\|x\|}{\|[x,H]\|^2}$ over open subsets of $\mathscr{O}(H_0)$ which will allow us to show that, over these open sets, $\|dZ_x\|$ is bounded and, consequently, that Z is Lipschitz.

Lemma 2.8. *There exists $C>0$ such that if $x\in\mathscr{O}(H_0)$ then $\|x\|>C$.*

Proof. The point is that in a semisimple Lie algebra an adjoint orbit $\mathscr{O}(X)$ is closed if $\mathrm{ad}(X)$ is diagonalizable. In particular, $\mathscr{O}(H_0)$ is closed and does not contain the origin. Therefore, $\mathscr{O}(H_0)$ does not approach 0 and it follows that $\inf_{x\in\mathscr{O}(H_0)}\|x\|>0$. □

The following lemma from linear algebra will be used to estimate $\|dZ_x\|$.

Lemma 2.9. *Let D_n and X_n be sequences of complex matrices such that*

(1) *Each D_n is diagonalizable and* $\lim D_n = \infty$.
(2) $\lim X_n = 0$.

Then there exists a subsequence n_k with $\lambda_{n_k}\in\mathbb{C}$ such that $\lim_k \lambda_{n_k} = \infty$ *e λ_{n_k} is an eigenvalue of $M_{n_k} = D_{n_k} + X_{n_k}$.*

Proof. Denote by a_n the diagonal entry of D_n that has the largest absolute value among all diagonal entries of D_n. Then $\lim a_n = \infty$, since $\lim D_n = \infty$. Consider the sequence

$$M_n = \frac{1}{a_n}(D_n + X_n).$$

We have $\lim \frac{1}{a_n}X_n = 0$. On the other hand, $\frac{1}{a_n}D_n$ is a bounded sequence, therefore there exists a subsequence n_k such that $\lim_k \frac{1}{a_{n_k}}D_{n_k} = D$. Consequently, $\lim_k \frac{1}{a_{n_k}}M_{n_k} = D$. We may refine the subsequence n_k such that the entry a_{n_k} of D_{n_k} occurs always at the same position for all k. Thus D is a diagonal matrix with 1 as an eigenvalue, since there exists a diagonal entry such that for all k, the entry of $\frac{1}{a_{n_k}}D_{n_k}$ in this position is 1.

The limit $\lim_k \frac{1}{a_{n_k}}M_{n_k} = D$ guarantees that for all $\varepsilon>0$ there exists $k_0\in\mathbb{N}$ such that if $k\ge k_0$ then $\frac{1}{a_{n_k}}M_{n_k}$ has an eigenvalue μ_{n_k} with $|\mu_{n_k}-1|<\varepsilon$. Setting $\varepsilon = 1/2$ we obtain $|\mu_{n_k}|>1/2$. Therefore, $\lambda_{n_k} = a_{n_k}\mu_{n_k}$ is an eigenvalue of M_{n_k} and $\lim\lambda_{n_k} = \infty$. □

The following lemma shows that the adjoint orbit $\mathscr{O}(H_0)$ is not asymptotic to the Cartan subalgebra $\mathfrak{h}$.

Lemma 2.10. *Let $\mathscr{O}(H_0)\cap\mathfrak{h}$ be the finite set of singularities of f_H in $\mathscr{O}(H_0)$. Given $\varepsilon>0$ denote by O_ε the set of $x\in\mathscr{O}(H_0)$ which are at a distance greater than ε of the singularities:*

$$O_\varepsilon = \{x\in\mathscr{O}(H_0) : \forall y\in\mathscr{O}(H_0)\cap\mathfrak{h},\ \left\|x-y\right\|>\varepsilon\}.$$

Denote by $p:\mathfrak{g}\to\sum_{\alpha\in\Pi}\mathfrak{g}_\alpha$ the projection given by the decomposition $\mathfrak{g} = \mathfrak{h}\oplus\sum_{\alpha\in\Pi}\mathfrak{g}_\alpha$. Then we have the following properties:

(1) *Given $\varepsilon>0$ there exists $\delta>0$ such that, if $x\in O_\varepsilon$, then $\left\|p(x)\right\|>\delta$.*

(2) *There exists a constant* $\Gamma_\varepsilon > 0$ *such that if* $x \in O_\varepsilon$ *then*

$$\frac{\|x - p(x)\|}{\|p(x)\|} < \Gamma_\varepsilon.$$

Proof. Both properties are proved by contradiction.

(1) Assume the statement is false. Then there exist $\varepsilon > 0$ and a sequence $y_n \in O_\varepsilon$ such that $\lim_n p(y_n) = 0$. Set $y_n = H_n + Y_n$, with $H_n \in \mathfrak{h}$ and $Y_n = p(y_n)$. The contradiction hypothesis guarantees that $\lim y_n = \infty$, since otherwise there would exist a subsequence y_{n_k} with $\lim_k y_{n_k} = y$. This implies that $\lim H_{n_k} = y$ given that $\lim Y_{n_k} = 0$. Since $\mathfrak{h}$ and $\mathcal{O}(H_0)$ are closed, it follows that $y \in \mathcal{O}(H_0) \cap \mathfrak{h}$, contradicting the fact that y_n does not approach $\mathcal{O}(H_0) \cap \mathfrak{h}$. Consequently, $\lim H_n = \infty$.

We may now apply lemma 2.9 by taking $D_n = \mathrm{ad}(H_n)$ and $X_n = \mathrm{ad}(Y_n)$. This shows that there exists a subsequence n_k such that $\mathrm{ad}(y_{n_k}) = D_{n_k} + X_{n_k}$ has an eigenvalue λ_{n_k} with $\lim \lambda_{n_k} = \infty$. But this is a contradiction because $y_n \in \mathcal{O}(H_0)$ and, therefore, the eigenvalues of $\mathrm{ad}(y_n)$ are the same as the eigenvalues of $\mathrm{ad}(H_0)$.

(2) Assume the statement is false. Then there exists a sequence $y_n \in O_\varepsilon$ such that $\lim \frac{\|y_n - p(y_n)\|}{\|p(y_n)\|} = \infty$. That is, $\lim \frac{\|p(y_n)\|}{\|y_n - p(y_n)\|} = 0$ or alternatively

$$\lim \frac{p(y_n)}{\|y_n - p(y_n)\|} = 0.$$

Set $H_n = y_n - p(y_n) \in \mathfrak{h}$, $D_n = \mathrm{ad}(H_n)$ and $X_n = \mathrm{ad}(p(y_n))$. As in the proof of lemma 2.9, let a_n be the eigenvalue of D_n with largest absolute value, so that $\|D_n\| = |a_n|$. Since the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is injective, there exist constants $C_1, C_2 > 0$ such that for all $Z \in \mathfrak{g}$ we have $C_1 \|\mathrm{ad}(Z)\| \ge \|Z\| \ge C_2 \|\mathrm{ad}(Z)\|$. In particular, $\|H_n\| \ge C_2 \|D_n\|$. Therefore,

$$\lim \frac{p(y_n)}{|a_n|} = 0$$

and we obtain

$$\lim \frac{X_n}{|a_n|} = 0.$$

Now, to arrive at a contradiction, we proceed as in the proof of lemma 2.9: there exists a subsequence n_k such that $\frac{1}{|a_{n_k}|}(D_{n_k} + X_{n_k})$ converges to a limit which has an eigenvalue equal to 1. Therefore, from a certain k_0 onwards, each $\frac{1}{|a_{n_k}|}(D_{n_k} + X_{n_k})$ has an eigenvalue with absolute value $> 1/2$, which implies that $\mathrm{ad}(y_{n_k}) = D_{n_k} + X_{n_k}$ has a sequence of eigenvalues that converges to ∞. However, as in item (1), this is a contradiction since $y_n \in \mathcal{O}(H_0)$ and, consequently, the eigenvalues of $\mathrm{ad}(y_n)$ are the same as those of $\mathrm{ad}(H_0)$. □

Now it is possible to show that $\|dZ_x\|$ is bounded in O_ε (and obviously $\|d(iZ)_x\|$ is bounded as well).

Lemma 2.11. *Given* $\varepsilon > 0$ *there exists* $L_\varepsilon > 0$ *such that* $\|dZ_x\| \le L_\varepsilon$ *if* $x \in O_\varepsilon$.

Proof. By lemma 2.7, we have

$$\|dZ_x\| \le M(\|\mathrm{ad}(H)\| + M\|H\|) \frac{\|x\|}{\|[x, H]\|^2}$$

if $x \notin \mathfrak{h}$. In particular, this inequality holds for $x \in O_\varepsilon$. Therefore, it suffices to estimate $\frac{\|x\|}{\|[x,H]\|^2}$.

Let $\delta > 0$ be given as item (1) of lemma 2.10, such that $\left\|p(x)\right\| > \delta$ if $x \in O_\varepsilon$. Since H is regular the restriction of $\mathrm{ad}(H)$ to $\sum_{\alpha\in\Pi} \mathfrak{g}_\alpha$ is an invertible linear map. Therefore, there exists $C > 0$ such that if $y \in \sum_{\alpha\in\Pi} \mathfrak{g}_\alpha$ and $\left\|y\right\| > \delta$, then $\left\|\mathrm{ad}(H)\, y\right\| > C\left\|y\right\|$. This implies that if $x \in O_\varepsilon$, then

$$\|[H,x]\| = \left\|[H, H' + p(x)]\right\| = \left\|[H, p(x)]\right\| > C\left\|p(x)\right\| > C\delta.$$

Consequently, choosing $\|[x,H]\| > C\delta$ as one of the factors of the denominator and $\|[x,H]\| > C\left\|p(x)\right\|$, it follows that

$$\frac{\|x\|}{\|[x,H]\|^2} < \frac{1}{C^2\delta} \cdot \frac{\|x\|}{\left\|p(x)\right\|}.$$

Now, $\|x\|^2 = \left\|x - p(x)\right\|^2 + \left\|p(x)\right\|^2$ since $x - p(x) \in \mathfrak{h}$ is orthogonal to $p(x) \in \sum_{\alpha\in\Pi} \mathfrak{g}_\alpha$. Therefore,

$$\begin{aligned}\left(\frac{\|x\|}{\left\|p(x)\right\|}\right)^2 &= \frac{\left\|x - p(x)\right\|^2 + \left\|p(x)\right\|^2}{\left\|p(x)\right\|^2} \\ &= \frac{\left\|x - p(x)\right\|^2}{\left\|p(x)\right\|^2} + 1.\end{aligned}$$

By lemma 2.10 (2), $\frac{\|x-p(x)\|^2}{\|p(x)\|^2} < \Gamma_\varepsilon^2$, so

$$\frac{\|x\|}{\left\|p(x)\right\|} < \sqrt{\Gamma_\varepsilon^2 + 1}$$

if $x \in O_\varepsilon$. This completes the proof, since

$$L_\varepsilon = \frac{M(\|\mathrm{ad}(H)\| + M\|H\|)}{C^2\delta}\sqrt{\Gamma_\varepsilon^2 + 1}$$

satisfies the desired inequality. □

A similar estimate shows that Z is bounded in each O_ε.

Lemma 2.12. *Given $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that $\|Z(x)\| \le M_\varepsilon$ if $x \in O_\varepsilon$.*

Proof. Let M be as in lemma 2.7. Then,

$$\begin{aligned}\|Z(x)\| &= \frac{1}{\|[x,H]\|^2}\|[x,[\tau x, H]]\| \\ &\le M\frac{\|x\| \cdot \|[x,H]\|}{\|[x,H]\|^2} = M\frac{\|x\|}{\|[x,H]\|}\end{aligned}$$

and, as in the proof of the previous lemma, $\frac{\|x\|}{\|[x,H]\|}$ in bounded on O_ε. □

Lemma 2.11 guarantees that Z is Lipschitz on O_ε with constant L_ε. The same is true for the vector field $e^{i\theta}Z$ with $\theta \in \mathbb{R}$ since $\left\|d\left(e^{i\theta}Z\right)\right\| = \|dZ\|$. By the previous lemma, $e^{i\theta}Z$ is bounded on O_ε. Combining these two facts, the theory of differential equations guarantees that all solutions of Z with initial condition $x(0) \in O_\varepsilon$ extend to a common interval of definition that contains 0.

Corollary 2.13. *Denote by ϕ_t^θ the local flow of the vector field $e^{i\theta}Z$. Then, given $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ such that $\phi_t^\theta(x)$ is well defined if $t \in (-\sigma_\varepsilon, \sigma_\varepsilon)$ and $x \in O_\varepsilon$. Under these conditions, $\phi_t^\theta(x) \in O_\varepsilon$.*

We are now ready to prove item (2) of theorem 2.2.

Proposition 2.14. *If $c_1, c_2 \in \mathbb{C}$ are regular values then the level manifolds $f_H^{-1}(c_1)$ and $f_H^{-1}(c_2)$ are diffeomorphic.*

Proof. On the set of regular values, define the equivalence relation $c_1 \sim c_2$ if $f_H^{-1}(c_1)$ and $f_H^{-1}(c_2)$ are diffeomorphic. We must show there exists a single equivalence class. To do so, it suffices to show that if $c \in \mathbb{C}$ is a regular value, then there exists a neighbourhood U of c such that for all $d \in U$, $f_H^{-1}(d)$ and $f_H^{-1}(c)$ are diffeomorphic. Indeed, this guarantees that the equivalence classes are open subsets (and, consequently, closed). However, the set of regular values is connected in $\mathbb{C}$ since it is the complement of a finite set.

Fix a regular value c. Since $f_H^{-1}(c)$ does not intercept the set of regular points, there exists $\varepsilon > 0$ such that $f^{-1}(c) \subset O_\varepsilon$.

Let σ_ε be as in corollary 2.13. Then $\phi_t^\theta(x)$ is defined for $t \in (-\sigma_\varepsilon, \sigma_\varepsilon)$ and $x \in O_\varepsilon$. In particular, it is also defined for $x \in f_H^{-1}(c)$. For a fixed x, the curve

$$\gamma_\theta : t \in (-\sigma_\varepsilon, \sigma_\varepsilon) \mapsto f_H\left(\phi_t^\theta(x)\right) \in \mathbb{C}$$

has derivative $\gamma_\theta'(t) = \left(df_H\right)_{\phi_t^\theta(x)}\left(e^{i\theta} Z\left(\phi_t^\theta(x)\right)\right)$. However, by definition of the field Z, $\left(df_H\right)_y\left(Z(y)\right) = 1$, so we have $\gamma_\theta'(t) = e^{i\theta}$. Therefore,

$$\begin{aligned} \gamma_\theta(t) &= \gamma_\theta(0) + \int_0^t \gamma_\theta'(s)\, ds \\ &= f_H(x) + te^{i\theta}. \end{aligned}$$

That is, $f_H\left(\phi_t^\theta(x)\right) = f_H(x) + te^{i\theta}$. In particular, if $x \in f_H^{-1}(c)$ then $\phi_t^\theta(x) = f_H^{-1}\left(c + te^{i\theta}\right)$, which means that $\phi_t^\theta\left(f_H^{-1}(c)\right) \subset f_H^{-1}\left(c + te^{i\theta}\right)$. The opposite inclusion is obtained applying the inverse flow ϕ_{-t}^θ, and we conclude that $\phi_t^\theta\left(f_H^{-1}(c)\right) = f_H^{-1}\left(c + te^{i\theta}\right)$. Thus, ϕ_t^θ is a diffeomorphism between $f_H^{-1}(c) = f_H^{-1}\left(c + te^{i\theta}\right)$.

This shows that every regular value in the open ball $B(c, \sigma_\varepsilon)$ is equivalent to c, that is, its fibre is diffeomorphic to the fibre at c. □

2.4. **Symplectic form.** The symplectic form that solves item (3) of theorem 2.2 is the imaginary part of the Hermitian form $\mathscr{H}$ from (2.1). We write the real and imaginary parts of $\mathscr{H}$ as

$$\mathscr{H}(X, Y) = (X, Y) + i\Omega(X, Y) \qquad X, Y \in \mathfrak{g}.$$

The real part $(\cdot, \cdot)$ is an inner product (since $(X, X) = \mathscr{H}(X, X)$) and the imaginary part of Ω is a symplectic form on $\mathfrak{g}$. Indeed, we have

$$0 \neq i\mathscr{H}(X, X) = \mathscr{H}(iX, X) = i\Omega(iX, X),$$

that is, $\Omega(iX, X) \neq 0$ for all $X \in \mathfrak{g}$, which shows that Ω is nondegenerate. Moreover, $d\Omega = 0$ because Ω is a constant bilinear form.

The fact that $\Omega(iX, X) \neq 0$ for all $X \in \mathfrak{g}$ guarantees that the restriction of Ω to any complex subspace of $\mathfrak{g}$ is also nondegenerate.

Now, the tangent spaces to $\mathcal{O}(H_0)$ are complex vector subspaces of $\mathfrak{g}$. Therefore, the pullback of Ω by the inclusion $\mathcal{O}(H_0) \hookrightarrow \mathfrak{g}$ defines a symplectic form on $\mathcal{O}(H_0)$.

Finally, the subspaces tangent to the level manifolds $f_H^{-1}(c)$ are complex subspaces of $\mathfrak{g}$ as well. Thus, if c is a regular value then $f_H^{-1}(c)$ is a symplectic submanifold of $\mathcal{O}(H_0)$.

This concludes the proof of item (3) of theorem 2.2.

Remark 2.15. An adjoint orbit $\mathcal{O}(X) \subset \mathfrak{g}$ admits another natural symplectic form ω besides the form Ω defined by $\mathscr{H}$. In fact, since $\mathfrak{g}$ is semisimple, the adjoint representation is isomorphic to the co-adjoint representation (via the Cartan–Killing form $\langle \cdot, \cdot \rangle$). Hence, the general construction of symplectic forms on co-adjoint orbits of Kirillov–Kostant–Souriaux can be carried through to the adjoint orbits of $\mathfrak{g}$. This yields the symplectic form ω on $\mathcal{O}(X)$ defined by $\omega_x([x, A], [x, B]) = \langle x, [A, B] \rangle$, where $x \in \mathcal{O}(X)$ and $A, B \in \mathfrak{g}$ (recall that $[x, A], [x, B] \in T_x\mathcal{O}(X)$). Nevertheless, the regular fibres $f_H^{-1}(c)$ of f_H are not symplectic submanifolds with respect to this ω. In fact, the vector $[x,H]$ is a tangent to $f_H^{-1}(c)$, since if $x\in f_H^{-1}(c)$, then

$$\left(df_H\right)_x([x,H])=\langle H,[x,H]\rangle=\langle [H,H],x\rangle=0.$$

If x is a regular point, then $[x,H]\neq 0$, but if $[x,A]$ (with $x\in\mathcal{O}(X)$ and $A\in\mathfrak{g}$) is tangent to $f_H^{-1}(c)$ then

$$\omega_x([x,H],[x,A])=\langle x,[H,A]\rangle=0$$

since $0=\left(df_H\right)_x([x,A])=\langle H,[A,x]\rangle=\langle x,[H,A]\rangle$.

Now a few comments about the singular fibres. First a note on the special case when $H_0\in\mathfrak{h}_{\mathbb{R}}$. Let wH_0, $w\in\mathcal{W}$, be a singularity. Define

$$\Pi(wH_0)=\{\alpha\in\Pi:\alpha(H_0)>0\}.$$

Then the subspaces

$$\mathfrak{n}^{\pm}(wH_0)=\sum_{\alpha\in\pm\Pi(wH_0)}\mathfrak{g}_\alpha$$

are the nilpotent subalgebras of $\mathfrak{g}$. Let $N^{\pm}(wH_0)$ be the connected groups with Lie algebra $\mathfrak{n}^{\pm}(wH_0)$. Then the following result holds true (see Helgason):

- The map $n\in N^+(wH_0)\mapsto \mathrm{Ad}(n)(wH_0)-wH_0\in\mathfrak{n}^+(wH_0)$ is a diffeomorphism. Similarly, there is such an isomorphism between $N^-(wH_0)$ and $\mathfrak{n}^-(wH_0)$.

In particular, this implies that for all $n\in N^{\pm}(wH_0)$, $\mathrm{Ad}(n)(wH_0)=wH_0+X$ with $X\in\mathfrak{n}^{\pm}$. Therefore,

$$f_H(\mathrm{Ad}(n)\,wH_0)=\langle H,wH_0+X\rangle=\langle H,wH_0\rangle=f_H(wH_0).$$

Consequently, the complex subspaces $\mathrm{Ad}\left(N^{\pm}(wH_0)\right)(wH_0)=(wH_0)+\mathfrak{n}^{\pm}(wH_0)$ are contained in the singular fibre $f_H^{-1}(\langle H,wH_0\rangle)$. This will be enough for us to analyse the singular fibre on the next example. For higher dimensions the structure of the singular fibres turns out rather more intricate, we will approach this issue in the forthcoming paper [GGS].

2.5. **Fukaya–Seidel cateogory for the $\mathfrak{sl}(2,\mathbb{C})$ orbit.** We now describe the Fukaya–Seidel category associated to the Landau–Ginzburg model obtained from theorem 2.2 by choosing in $\mathfrak{sl}(2,\mathbb{C})$ the elements

$$H=H_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Hence $\mathcal{O}(H_0)$ is the set of matrices in $\mathfrak{sl}(2,\mathbb{C})$ with eigenvalues ±1. This set forms a submanifold Σ of $\mathfrak{sl}(2,\mathbb{C})$ of real dimension 4 (a complex surface). In this case the Weyl group is $\mathcal{W}=\{\pm1\}$. Therefore, the potential $f_H=:\Sigma\to\mathbb{C}$ has two singularities, namely $\pm H$. We obtain:

Example 2.16. The Fukaya–Seidel category of (Σ,f_H) with integer coefficients is generated by 2 Lagrangians L_0 and L_1 in degrees 0 and 1 respectively, with morphisms:

$$\mathrm{Hom}(L_0,L_1)=\mathbb{Z}^2,\qquad \mathrm{Hom}(L_0,L_0)=\mathrm{Hom}(L_1,L_1)=\mathbb{Z},\qquad \mathrm{Hom}(L_1,L_0)=0$$

and the products m_k all vanish except for $m_2(\cdot,id)$ and $m_2(id,\cdot)$.

The regular fibres are submanifolds of real dimension 2 (complex curves).

With respect to the singular fibres, for example $F_H^{-1}(\langle H,H\rangle)$, we have the following data:

$$\mathfrak{n}^+(H)=\{\begin{pmatrix}0&z\\0&0\end{pmatrix}:z\in\mathbb{C}\}$$

whereas $\mathfrak{n}^-(H)$ are lower triangular. Then, the sets

$$H+\mathfrak{n}^+(H)=\{\begin{pmatrix}1&z\\0&-1\end{pmatrix}:z\in\mathbb{C}\}\qquad H+\mathfrak{n}^-(H)=\{\begin{pmatrix}1&0\\z&-1\end{pmatrix}:z\in\mathbb{C}\}$$

are contained in $f_H^{-1}(\langle H,H\rangle)$. Counting dimensions we conclude that the union of these two affine subspaces is exactly $f_H^{-1}(\langle H,H\rangle)$. More precisely: the matrix

$$X=\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\in\mathfrak{sl}(2,\mathbb{C})$$

belongs to $f_H^{-1}(\langle H,H\rangle)$ if and only if $\operatorname{tr}(XH)=\operatorname{tr}\left(H^2\right)=2$, since in this case there exists a constant c such that for all $A,B\in\mathfrak{sl}(2,\mathbb{C})$, $\langle A,B\rangle = c\operatorname{tr}(AB)$. Since $\operatorname{tr}(XH)=2a$, it follows that $\operatorname{tr}(XH)=2$ if and only if $a=1$. Then, the characteristic polynomial of X is

$$p_X(\lambda)=\lambda^2-(1+bc).$$

Since $X\in\mathscr{O}(H)$, it has eigenvalues ±1. This happens if and only if $1+bc=1$, that is, $bc=0$. Therefore, $X\in f_H^{-1}(\langle H,H\rangle)$ if and only if

$$X=\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}\qquad\text{or}\qquad X=\begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}.$$

We can also describe the regular fibres. For example, the matrix

$$X=\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\in\mathfrak{sl}(2,\mathbb{C})$$

belongs to the regular fibre $f_H^{-1}(0)$ if and only if $2a=\operatorname{tr}(XH)=0$, that is, $a=0$. Hence , the characteristic polynomial is $p_X(\lambda)=\lambda^2-bc$ and the eigenvalues are ±1 if and only if $bc=1$. Therefore, $f_H^{-1}(0)$ consists of matrices

$$\begin{pmatrix} 0 & b \\ 1/b & 0 \end{pmatrix}\qquad 0\neq b\in\mathbb{C}.$$

Thus, we find that $f_H^{-1}(0)$ and all regular fibres are homeomorphic to the cylinder $\mathbb{C}\setminus\{0\}$.

Now we will describe the thimbles using branched covers. We have the surface $\Sigma=\{x^2+yz=1\}$ together with the potential

$$f_H\colon\Sigma\to\mathbb{C}$$
$$(x,y,z)\mapsto 2x.$$

To find the critical points of $f_H|_\Sigma$ we use Lagrange multipliers, thus solving $\operatorname{grad} f=\xi\operatorname{grad} g$ with $g=1$ which gives $(2,0,0)=(2x,z,y)$, where $g=g(x,y,z)=x^2+yz$. We obtain the critical point $(x,y,z)=(1,0,0)$ with corresponding singular fibre $f_H^{-1}(1)=\{yz=0\}$.

On the other hand, for a regular value $\lambda\in\mathbb{C}$ we write $2x=\lambda$ that is $x=\lambda/2$, so $\frac{\lambda^2}{4}+yz=1$. We set

$$\Sigma_\lambda:=\left\{yz=1-\frac{\lambda^2}{4}\right\}.$$

We first consider the cut given by $y=z$ where we need to analyse the two branches of the square root $y=\pm\sqrt{1-\frac{\lambda^2}{4}}$. We get the two curves

$$\left(\frac{\lambda}{2},\pm\sqrt{1-\frac{\lambda^2}{4}},\pm\sqrt{1-\frac{\lambda^2}{4}}\right)\xrightarrow{\lambda\to2}(1,0,0).$$

Using these curves we want to write down the thimbles, that is, for each λ we wish to identify a circle in X with $\gamma(t)$ with

$\gamma(0)=\left(\frac{\lambda}{2},\sqrt{1-\frac{\lambda^2}{4}},\sqrt{1-\frac{\lambda^2}{4}}\right)$ and $\gamma(\pi)=\left(\frac{\lambda}{2},-\sqrt{1-\frac{\lambda^2}{4}},-\sqrt{1-\frac{\lambda^2}{4}}\right)$. For $0\le t\le2\pi$ we take the thimble to be:

$$\alpha_\lambda(t)=\left(\frac{\lambda}{2},e^{it}\sqrt{1-\frac{\lambda^2}{4}},e^{-it}\sqrt{1-\frac{\lambda^2}{4}}\right).$$

Thus, $\alpha_\lambda(t) \to (1,0,0)$ as $\lambda \to 2$ and for a regular value λ the curve $\gamma(t) := \alpha_\lambda(t)$ is a Lagrangian circle on the fibre $f_H^{-1}(\lambda)$. We fix a regular value, say $0 \in \mathbb{C}$, and consider the straight line joining the regular value 0 to the critical value 2 (that is, a matching path). Then the family of Lagrangian circles $\alpha_\lambda(t)$ is fibred over the straight line and produces the Lagrangian thimble. With a similar analysis we can produce the Lefschetz thimble associated to the critical value -2.

Considering the line joining the two critical values -2 and 2 together with the union of the two corresponding Lefcshetz thimbles we obtain a sphere Y in the orbit $\Sigma = \mathcal{O}(H_0)$. The next result shows that this sphere is a Lagrangean subvariety of Σ.

Lemma 2.17. *Consider the orbit Σ with the symplectic form Ω as in section 2.4, then $Y \subset \Sigma$ given by the equation $x^2 + y^2 + z^2 = 1$ is a Lagrangean submanifold.*

Proof. Let $\mathfrak{u}$ be a real compact form of $\mathfrak{sl}(2,\mathbb{C})$. Here $\mathfrak{u}$ is the set of anti-Hermitian matrices with trace zero, thus $\sqrt{-1}\mathfrak{u}$ is the set of Hermitian matrices with trace zero. Note that the submanifold Y can be described as the intersection$Y = \Sigma \cap \sqrt{-1}\mathfrak{u}$. In fact, an arbitrary matrix $S \in \sqrt{-1}\mathfrak{u}$ has the form

$$S = \begin{pmatrix} r & -p+iq \\ -p-iq & -r \end{pmatrix},$$

with $p, q, r \in \mathbb{R}$. Since the orbit Σ consists of 2×2 complex matrices whose entries satisfy $x^2 + yz = 1$, we see that $S \in \Sigma$ if and only if its entries satisfy $r^2 + p^2 + q^2 = 1$.

The tangent space of Y at S is given by $T_S Y = \{[S,A]; A \in \mathfrak{u}\}$. Since $[\sqrt{-1}\mathfrak{u}, \mathfrak{u}] \subset \sqrt{-1}\mathfrak{u}$ and $tr(M.N)$ is real when $M, N \in \sqrt{-1}\mathfrak{u}$, we conclude that $\Omega_S([S,A],[S,B]) = 0$ thus Y is Lagrangean. □

Remark 2.18. In greater generality, let $\mathfrak{g}$ be a simple complex Lie algebra and $\mathfrak{u}$ a real compact form of $\mathfrak{g}$. Consider an adjoint orbit $\mathcal{O}(H_0)$. It is known that the intersection $\mathcal{O}(H_0) \cap \sqrt{-1}\mathfrak{u}$ is a generalized flag variety. An argument similar to the previous one shows that such generalized flag varieties are Lagrangeans inside the corresponding orbits with respect to the symplectic form Ω.

Remark 2.19. [GGS] we take an appropriate choice of symplectic structure on adjoint orbits for which each adjoint orbit of a semisimple Lie group becomes symplectomorphic to the cotangent bundle of a generalized flag variety. In this particular example of $\mathfrak{sl}(2,\mathbb{C})$ the flag variety is $\mathbb{CP}^1 \approx S^2$ and consequently $\mathcal{O}(H_0) \approx T^*\mathbb{CP}^1$. See Section 3 bellow for further details.

Remark 2.20. The symplectic topology of the Milnor fibration with singularity of type A_n was studied in [KS] using braid group techniques. In particular one can read off the Floer cohomology of $T^*(S^2)$ considered with the standard symplectic structure. Our construction of the adjoint orbit for $\mathfrak{sl}(2,\mathbb{C})$ endows $T^*(S^2)$ with another symplectic structure, and our calculations use completely different techniques. This coincidence of examples is a feature of low dimensions, and will not repeat itself for the orbits of $\mathfrak{sl}(n,\mathbb{C})$ with $n > 2$ where our flag varieties are not spheres.

We will now describe the Fukaya–Seidel category associated to the Landau–Ginzburg model $LG(\Sigma, f_H)$, whose objects are the vanishing cycles (or Lagrangian thimbles). We first recall the definition.

Definition 2.21. ([AKO1], def. 3.1) The *directed category of vanishing cycles* $\mathrm{Lag}_{\mathrm{vc}}(f,\gamma)$ is an A_∞-category (over a coefficient ring R) with r objects $L_1, \ldots, L_r$ corresponding to the vanishing cycles (or more accurately, to the thimbles); the morphisms between the objects are given by

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j, R) = R^{|L_i \cap L_j|} & \text{if } \; i < j \\ R \cdot \mathrm{id} & \text{if } \; i = j \\ 0 & \text{if } \; i > j \end{cases} \tag{2.5}$$

and the differential m_1, composition m_2 and higher order products m_k are defined in terms of Lagrangian Floer homology inside the regular fibre Σ_0. More precisely,

$$m_k\colon \operatorname{Hom}(L_{i_0}, L_{i_1}) \otimes \cdots \otimes \operatorname{Hom}(L_{i_{k-1}}, L_{i_k}) \to \operatorname{Hom}(L_{i_0}, L_{i_k})[2-k]$$

is trivial when the inequality $i_0 < i_1 < \cdots < i_k$ fails to hold. When $i_0 < \cdots < i_k$, m_k is defined by fixing a generic ω-compatible almost-complex structure on Σ_0 and counting pseudo-holomorphic maps from a disc with $k+1$ cyclically ordered marked points on its boundary to Σ_0, mapping the marked points to the given intersection points between vanishing cycles, and the portions of boundary between them to $L_{i_0}, \ldots, L_{i_k}$ respectively. We refer to this as the *Fukaya–Seidel category*.

To proceed with our example, we fix the regular value $0 \in \mathbb{C}$ of our LG model and consider the line segments β and γ that join -2 to 0 and 0 to 2, respectively. The objects of the Fukaya–Seidel category are the two Lagrangean thimbles $L_0 := \alpha_{\beta(s)}(t)$ and $\tilde{L}_0 := \alpha_{\gamma(s)}(t)$ (abusing notation we consider as L_0 and $\tilde{L}_0$ only the vanishing cycles in the regular fiber Σ_0; in our case, both a circle S^1).

Remark 2.22. A different choice of path joining the critical values to the regular value will result in an equivalent category, see [Se].

To specify the products in the category, we need to describe $HF^*(L_0, \tilde{L}_0)$. However, as Floer cohomology is rather difficult to calculate, we will use an indirect calculation allowing as to connect these Floer groups to the de Rham cohomology of S^1 (lemma 2.25 below).

First notice that in our case the regular fiber is homeomorphic to $\mathbb{C}^*$, which can be identified with the cylinder T^*S^1 via the map $g : \mathbb{C}^* \to T^*S^1$ given by

$$g(y) = \left(\frac{y}{|y|}, \ln|y|\right). \tag{2.6}$$

On the regular fiber Σ_0, the vanishing cycles coincide with the curve $(0, e^{it}, e^{-it}) \in \Sigma_0$ (just make $\lambda = 0$ in the above expressions for the thimbles).

We now observe a delicate issue: the regular fibre $\mathbb{C}^*$ inherits the symplectic structure Ω from the adjoint orbit. Such symplectic structure is (up to a constant) the canonical Kähler structure of $\mathbb{C}^*$ regarded as a submanifold of $\mathbb{C}$. Via 2.5 we regard the regular fibre as (T^*S^1, Ω) which, however, is not symplectomorphic to (T^*S^1, ω_c), where ω_c is the canonical exact symplectic form on the cotangent bundle, see [EG]. Nevertheless, thm. 2.23 below makes it is possible to use the canonical symplectic form ω_c to help find the required Floer cohomology.

Recall that a Lagrangian submanifold L of $(X, \omega = d\theta)$ is called *admissible* provided L is exact (that is, $[\theta|L] = 0$), spin, and has zero Maslov class.

By Weinstein's tubular neighborhood theorem, there exists a symplectic embedding κ from a tubular neighborhood of $S^1 \subset (T^*S^1, \omega_c)$ into (T^*S^1, Ω) such that $\kappa(S^1) = S^1$ (note que S^1 is Lagrangean with respect to both symplectic structures in T^*S^1). The next result relates the Floer homologies via the map κ.

Theorem 2.23 ([FSS], Lemma 8)**.** *Let $(X, \omega = d\theta)$ be an exact symplectic manifold and N a Lagrangean submanifold of X. Let κ be the symplectic embedding given by the theorem of Weinstein from a neighborhood $V(N)$ of N in T^*N to X. Let $L_0, L_1 \subset V(N)$ be closed admissible Lagrangean submanifolds. Then $HF^*(\kappa(L_0), \kappa(L_1)) \cong HF^*(L_0, L_1)$.*

Observe that the Floer cohomology on the lhs takes place in X whereas on the rhs it takes place in T^*N.

Remark 2.24. In [FSS] thm. 2.23 appears in the context of Lefschetz fibrations with a real structure, however, the real structure is not used in its proof, thus the result applies to our situation.

Returning to our example, we now consider the cotangent bundle (T^*S^1, ω_c) with its canonical symplectic form. To find the Floer homology, we will perturb the circle $\tilde{L}_0$ by Hamiltonian isotopy as follows: let $f : S^1 \to \mathbb{R}$ be a Morse function and $\epsilon > 0$ small. Let

$$L_1 := \{\text{ graph of the exact 1-form } \epsilon df\}.$$

We have that L_1 is a Hamiltonian isotopic image of $\tilde{L}_0$ (with isotopy given by $H = \epsilon f \circ \pi$, where $\pi : T^*S^1 \to S^1$ is the canonical projection) and L_0 intersects transversally L_1 at the critical points of f. The next result is well known and relates the Floer homology $HF(L_0, \tilde{L}_0)$ with the Morse homology of f (keeping in mind that Floer homology is invariant by Hamiltonian isotopies), see [Au] and [FOOO].

Lemma 2.25. $HF^*(L_0, L_1) \approx H^*(S^1; \mathbb{R})$.

Combining lemma 2.25 and theorem 2.23 we obtain:

Corollary 2.26. *For L_0 and L_1 considered as Lagrangians in (Σ_0, Ω) we have $HF^*(L_0, L_1) \approx H^*(S^1; \mathbb{R})$.*

We now fix a Morse function $f : S^1 \to \mathbb{R}$ with exactly 2 critical points. Since the product m_1 in the Fukaya–Seidel category is the differential of Floer homology, using lemma 2.25, we obtain the following description of the products m_i:

Lemma 2.27. *The products m_i for the Fukaya–Seidel category of $LG(X, W)$ all vanish, except for the trivial products $m_2(id, \cdot)$ and $m_2(\cdot, id)$.*

Explicit calculation (see [Au], [FOOO]) shows that a critical point of f (which results in an intersection of the Lagrangeans) with Morse index $i(p)$ defines a generator of degree $deg(p) = n - i(p)$ in the Floer complex, where n is the dimension of the variety (in our case $\dim S^1 = 1$). Since we have chosen f with exactly two critical points (a maximum and a minimum), the Morse indices are 0 and 1, respectively. We obtain:

Lemma 2.28. *There is a natural choice of grading such that $deg(L_0) = 0$ and $deg(L_1) = 1$.*

Remark 2.29. Comparison with the AKO-mirror of $\mathbb{CP}^1$: We observe that, despite the isomorphism $\Sigma \simeq T^*\mathbb{CP}^1$ the Fukaya–Seidel category we just described is not isomorphic to the Fukaya–Seidel category of the mirror of $\mathbb{CP}^1$ described in [AKO1]. Indeed, although the number of objects, morphisms and products of the A_∞ structures coincide, the gradings are different. It is an open question to determine which complex (algebraic) variety has the the Landau–Ginzburg model $LG(\Sigma, f_H)$ we have described as its mirror.

3. Topology of the regular fibres

To describe the regular fibres of f_H we use another description of the adjoint orbit, namely we regard it as a vector bundle. In fact, the adjoint orbit has various realizations (e.g. as a homogeneous space, and as the cotangent bundle of a flag manifold). These various realizations, as well as their symplectic geometry, are explored in detail in [GGS]. The realization of the orbit as a cotangent bundle appeared earlier in [ABB].

To study the topology of the regular fibres, we first identify the orbit $\mathcal{O}(H_0)$ with the cotangent bundle of a flag manifold. Here is a summary of the construction. Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h}$. The adjoint orbit of an element $H_0 \subset \mathfrak{h}$ can be identified with the homogeneous space G/Z_{H_0}, where Z_{H_0} is the centraliser of H_0 in G. We also identify the adjoint orbit $\mathrm{Ad}(K) \cdot H_0$ of the maximal compact subgroup K of G with the flag manifold $\mathbb{F}_{H_0} = G/P_{H_0}$, where P_{H_0} is the parabolic subgroup which contains Z_{H_0}. Using the construction of the vector bundle associated to the P_{H_0}-principal bundle $G \to \mathbb{F}_{H_0} = G/P_{H_0}$ we showed that the quotient G/Z_{H_0} has the structure of a vector bundle over $\mathbb{F}_{H_0}$ isomorphic to the cotangent bundle $T^*\mathbb{F}_{H_0}$ [GGS, thm. 2.1].

Remark 3.1. In Example 2.21, the associated flag variety is $\mathbb{CP}^1 \approx S^2$ and consequently $\mathscr{O}(H_0) \approx T^*\mathbb{CP}^1$.

We now use the identification of the orbit with the cotangent bundle of a flag to describe the regular fibres of f_H. Our height function $f_H(x) = \langle H, x\rangle$, $x \in \mathscr{O}(H_0)$, takes values in $\mathbb{C}$, whereas, by hypothesis, H and H_0 are real, that is, belong to $\mathfrak{h}_\mathbb{R}$, and H is regular. We showed in proposition 2.4 that f_H has a finite number of singularities. These singular points belong to $\mathbb{F}_{H_0}$, regarded as the orbit of the compact group $U \cdot H_0$.

Since H and H_0 are real, f_H restricted to $\mathbb{F}_{H_0}$ takes real values. H and H_0 can be chosen in *general position* such that $\langle H, wH_0\rangle = \langle H, uH_0\rangle$ if and only if $w = u$, where $w, u \in \mathscr{W}$. (The latter condition implies that the singular levels do not intersect. Such general position may be obtained by fixing H_0 then varying H.)

In this section and the next, when we use the identification of the adjoint orbit with the cotangent bundle of a flag manifold, the word fibre appears in two senses: a fibre of the Lefschetz fibration f_H which is topologically nontrivial, and a fibre of the cotangent bundle $T^*\mathbb{F}_{H_0}$ which is a vector space. To avoid confusion between the two meanings of fibre, we introduce the term level:

Definition 3.2. We call $L(\xi) = f_H^{-1}\left(f_H(\xi)\right)$ the *level* of f_H passing through $\xi \in \mathscr{O}(H_0)$. If $L(\xi)$ contains a singularity of f_H we call it a *singular level*, otherwise we call it a *regular level.*

Notation 3.3. $\widetilde{X}$ denotes the vector field on $\mathbb{F}_{H_0}$ induced by $X \in \mathfrak{g}$, defined as $\widetilde{X}(x) = \frac{d}{dt} e^{tX} x_{|t=0}$.

Theorem 3.4. *A regular level $L(\xi)$ is an affine subbundle of the cotangent bundle restricted to the complement of the singular points $\mathbb{F}_{H_0} \setminus \mathscr{W} \cdot H_0$. More precisely, a regular level $L(\xi)$ surjects over $\mathbb{F}_{H_0} \setminus \mathscr{W} \cdot H_0$ and its intersection with the cotangent fibre $T_x^*\mathbb{F}_{H_0}$ is an affine subspace, whose underlying vector space is*

$$V_H(x) = \{\mu \in T_x^*\mathbb{F}_{H_0} : \mu\left(\widetilde{H}(x)\right) = 0\}.$$

Identifying $T^*\mathbb{F}_{H_0}$ with the tangent bundle $T\mathbb{F}_{H_0}$ via the Borel metric, the subspace $V_H(x)$ becomes the subspace orthogonal to $\widetilde{H}(x)$, which is exactly the space tangent to the level x of the function f_H restricted to the flag.

The proof of theorem 3.4 is a rather immediate consequence of the construction of the action of G on $T^*\mathbb{F}_{H_0}$, that identifies it with the adjoint orbit $\mathscr{O}(H_0) = \mathrm{Ad}(G) \cdot H_0$. It involves the following facts:

(1) The *real part* of f_H is known. In fact, let $\mathfrak{g}^R$ be the realification of $\mathfrak{g}$ (which is also a semisimplesimple Lie algebra). Denote by $\langle\cdot,\cdot\rangle^R$ the Cartan–Killing form of $\mathfrak{g}^R$. Then, $\langle\cdot,\cdot\rangle^R = 2\mathrm{Re}\langle\cdot,\cdot\rangle$. Thus, $\left(\mathrm{Re} f_H\right)(x) = 1/2 f^R(x)$ where $f^R(x) = \langle H, x\rangle^R$.
(2) The Cartan decomposition of $\mathfrak{g}$ (or rather of $\mathfrak{g}^R$) is given by $\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}$ where $\mathfrak{u}$ is the real compact form of $\mathfrak{g}$ and $\mathfrak{s} = i\mathfrak{u}$. The group $U = \langle \exp \mathfrak{u}\rangle$ is compact. The exponential is taken to any group G with Lie algebra $\mathfrak{g}$.
(3) Since $\mathfrak{u}$ is a real compact form, it follows that the restriction of the Cartan-Killing form $\langle\cdot,\cdot\rangle$ to $\mathfrak{u}$ is negative definite (and takes real values). Hence, the restriction to $i\mathfrak{u}$ is positive definite. Moreover, if $X \in \mathfrak{u}$ and $Y \in i\mathfrak{u}$ then $\langle X, Y\rangle$ is purely imaginary.
(4) The intersection $\mathscr{O}(H_0) \cap i\mathfrak{u}$ coincides with the flag $\mathbb{F}_{H_0} = \mathrm{Ad}(U) H_0$.
(5) The restriciton of f_H to $\mathscr{O}(H_0) \cap i\mathfrak{u} = \mathbb{F}_{H_0}$ is real, equal to $1/2 f^R$.
(6) The *imaginary part* of f_H comes from $f_{iH}(x) = \langle iH, x\rangle$, $x \in \mathscr{O}(H_0)$, in the following way:
$$f_{iH}(x) = i\langle H, x\rangle = i f_H(x) = -\mathrm{Im} f_H(x) + i \mathrm{Re} f_H(x),$$
therefore $\mathrm{Im} f_H(x) = -\mathrm{Re} f_{iH}(x) = -\mathrm{Re}\langle iH, x\rangle = -\frac{1}{2}\langle iH, x\rangle^R$. Hence,
$$f_H = f_H^R - i f_{iH}^R$$

where the upper index indicates that the height function is taken with respect to the real Cartan–Killing form $\langle\cdot,\cdot\rangle^R = 2\mathrm{Re}\langle\cdot,\cdot\rangle$. This seemingly trivial formula is useful to express f_H when we regard $\mathcal{O}(H_0)$ as $T^*\mathbb{F}_{H_0}$.

(7) *Height function on the cotangent bundle (real part):* If $X \in \mathfrak{s} = i\mathfrak{u}$ then $\alpha(X) = X^\# + V_X$. This means that the vector field $\overrightarrow{X}$ induced by X on $\mathcal{O}(H_0)$ is the Hamiltonian vector field of the function $\left(\widetilde{X},\cdot\right)_B + F_X^R$ where $(\cdot,\cdot)_B$ is the Borel metric on $\mathbb{F}_{H_0}$, $F_X^R = f_X^R \circ \pi$ and $\widetilde{X}$ is the vector field induced by X on $\mathbb{F}_{H_0}$.

In particular, the hypothesis that H is real implies that $H \in \mathfrak{s} = i\mathfrak{u}$ and therefore the vector field $\overrightarrow{H}$ induced by H on $\mathcal{O}(H_0)$ is the Hamiltonian of the function $\left(\widetilde{H},\cdot\right)_B + F_H^R$. On the other hand, we know that the vector field $\overrightarrow{H}$ (given by $\overrightarrow{H}(x) = [H,x]$) is the Hamiltonian of the function $f_H^R(x) = \langle H,x\rangle^R$ defined on $\mathcal{O}(H_0)$. Thus, the two functions give rise to the same Hamiltonian fields and consequently differ by a constant. That is, via the diffeomorphism between $\mathcal{O}(H_0)$ and $T^*\mathbb{F}_{H_0}$ the function $f_H^R(x) = \langle x,H\rangle$ is given by $f_H^R = \left(\widetilde{H},\cdot\right)_B + F_H^R + \mathrm{ct.}$

(8) *Height function on the cotangent bundle (imaginary part):* the imaginary part is given by f_{iH}^R. The difference here is that $iH \in \mathfrak{u}$, therefore $\overrightarrow{iH}$ is the Hamiltonian field of the function $\left(\widetilde{iH},\cdot\right)_B$. But $\overrightarrow{iH}$ is the Hamiltonian field of f_{iH}^R as well, thus $f_{iH}^R = \left(\widetilde{iH},\cdot\right)_B + \mathrm{ct}$. Together with the previous item, this gives

$$f_H = \left(\widetilde{H},\cdot\right)_B + F_H^R - i\left(\widetilde{iH},\cdot\right)_B + \mathrm{ct.}$$

(9) The constant of the previous item is calculated evaluating the equality on H_0; terms involving the Borel metric vanish (zero section). Therefore

$$\mathrm{ct} = f_H(H_0) - F_H^R(H_0) = f_H(H_0) - f_H^R(H_0) = \langle H,H_0\rangle - \langle H,H_0\rangle^R = 0$$

since $\langle H,H_0\rangle$ is real.

Proof of theorem 3.4: Choose a regular point $x \in \mathbb{F}_{H_0} = \mathcal{O}(H_0) \cap i\mathfrak{u}$. Then, the restriction of f_H to the tangent space $T_x\mathbb{F}_{H_0}$ (identified with $T_x^*\mathbb{F}_{H_0}$ by the Borel metric) is given by

$$\left(\widetilde{H}(x),\cdot\right)_B - i\left(\widetilde{iH}(x),\cdot\right)_B + f_H^R(x)$$

which is an affine map, hence surjective. So, if $x \in \mathbb{F}_{H_0}$ is a regular point of f_H (that is, $x \in \mathbb{F}_{H_0} \setminus \mathcal{W}\cdot H_0$) then every level of f_H intercepts $T_x\mathbb{F}_{H_0}$. This shows that every regular level $L(\xi)$ projects surjectivelly onto $\mathbb{F}_{H_0} \setminus \mathcal{W}\cdot H_0$. On the other hand, the intersection of a level $L(\xi)$ with the tangent space $T_x\mathbb{F}_{H_0}$ is given by the codimension 2 affine subspace

$$L(\xi) \cap T_x\mathbb{F}_{H_0} = \{v \in T_x\mathbb{F}_{H_0} : \left(\widetilde{H}(x),v\right)_B - i\left(\widetilde{iH}(x),v\right)_B = f_H^R(x) + f_H(\xi)\}$$

which shows that $L(\xi)$ is an affine subbundle of $T^*\mathbb{F}_{H_0}$. □

As a consequence we identify the topology of a regular level $L(\xi)$:

Corollary 3.5. *The homology of a regular level $L(\xi)$ coincides with that of $\mathbb{F}_{H_0} \setminus \mathcal{W}\cdot H_0$. In particular, the middle Betti number of $L(\xi)$ equals $k-1$, where k is the number of singularities of the fibration f_H (and equals the number of elements in the orbit $\mathcal{W}\cdot H_0$).*

4. Topology of the singular fibres

The singular levels of f_H are the levels that pass through wH_0, $w \in \mathcal{W}$. Assume that H_0 and H are in "general position", so that each singular fibre contains just one singularity.

The following proposition gives a description of the singular levels of f_H. In the statement, $\pi\colon \mathcal{O}(H_0) \to \mathbb{F}_{H_0}$ is the canonical projection that makes $\mathcal{O}(H_0) \approx T^*\mathbb{F}_{H_0}$, where $T^*\mathbb{F}_{H_0}$ is the flag manifold defined by H_0.

Proposition 4.1. *The singular fibre of $f_H^{-1}\left(f_H(wH_0)\right)$ passing through wH_0 is the disjoint union of the following sets:*

(1) *An affine subbundle of real codimension 2 of* $\mathcal{O}(H_0) \to \mathbb{F}_{H_0} \setminus \{uH_0 : u \in \mathcal{W}\}$ *over the set of regular points of* $\mathbb{F}_{H_0}$.
(2) *The fibre* $\pi^{-1}(wH_0)$. *As a subset of* $\mathfrak{g}$ *(in the adjoint orbit) this fibre is given by the affine subspace*

$$wH_0 + \mathfrak{n}^+(wH_0)$$

where $\mathfrak{n}_w^+$ *is the sum of eigenspaces with positive eigenvalues of* $\mathrm{ad}(wH_0)$.

The subspace $\mathfrak{n}^+(wH_0)$ in the statement is a nilpotent subalgebra given by

$$\mathfrak{n}^+(wH_0) = \sum_{\alpha \in \Pi(wH_0)} \mathfrak{g}_\alpha$$

where $\Pi(wH_0) = \{\alpha \in \Pi : \alpha(H_0) > 0\}$.

Proof. To prove the proposition we examine the intersection of the level $f_H^{-1}(f_H(wH_0))$ with the fibres of $\pi\colon \mathcal{O}(H_0) \to \mathbb{F}_{H_0}$. Such intersections can be described as follows:

(1) Let $x \in \mathbb{F}_{H_0}$ be a regular point of f_H, that is, $x \neq uH_0$ for all $u \in \mathcal{W}$. Then, the restriction of f_H to the cotangent fibre $\pi^{-1}\{x\}$ is an affine map, whose linear part is nonzero. Such linear part is the functional $\left(\widetilde{H}, \cdot\right)_B - i\left(\widetilde{iH}, \cdot\right)_B$, where $(\cdot,\cdot)_B$ is the Borel metric). If $x \in \mathbb{F}_{H_0}$ is a regular point, then the linear part has no zeros. This implies that all levels of f_H intersect $\pi^{-1}\{x\} = T_x^*\mathbb{F}_{H_0}$ on affine subspaces of complex codimension 1, proving statement (1).
(2) Let $N^+(wH_0)$ be the connected group with Lie algebra $\mathfrak{n}^+(wH_0)$. Then, the map

$$n \in N^+(wH_0) \mapsto \mathrm{Ad}(n)(wH_0) - wH_0 \in \mathfrak{n}^+(wH_0)$$

is a diffeomorphism. In particular, for all $n \in N^+(wH_0)$, $\mathrm{Ad}(n)(wH_0) = wH_0 + X$ with $X \in \mathfrak{n}^+$. Therefore,

$$f_H(\mathrm{Ad}(n)\, wH_0) = \langle H, wH_0 + X\rangle = \langle H, wH_0\rangle = f_H(wH_0). \tag{4.1}$$

Hence, the affine subspace $\mathrm{Ad}\left(N^+(wH_0)\right)(wH_0) = (wH_0) + \mathfrak{n}^+(wH_0)$ is contained in the singular level $f_H^{-1}(\langle H, wH_0\rangle)$.

Using the isomorphism $\mathcal{O}(H_0) \approx T^*\mathbb{F}_{H_0}$, we see that the fibre over wH_0 is precisely $(wH_0) + \mathfrak{n}^+(wH_0)$, proving statement (2).
(3) It remains to verify that if $uH_0 \neq wH_0$ then the fibre $\pi^{-1}\{uH_0\}$ does not intersect the level $f_H^{-1}(\langle H, wH_0\rangle)$. By the same argument as in the previous item, the fibre $\pi^{-1}\{uH_0\}$ in the adjoint orbit, is given by the adjoint subspace $(uH_0) + \mathfrak{n}^+(uH_0)$. By equalities (4.1) f_H is constant on this subspace and equals $f_H(uH_0)$. Since by hypothesis each singular level contains just one singularity, this shows that $f_H^{-1}(\langle H, wH_0\rangle)$ does not intersect the fibre over $uH_0 \neq wH_0$.

□

Corollary 4.2. *The homology of a singular level* $L(wH_0)$, $w \in \mathcal{W}$ *coincides with that of*

$$\mathbb{F}_{H_0} \setminus \{uH_0 \in \mathcal{W} \cdot H_0 : u \neq w\}.$$

In particular, the middle Betti number of $L(wH_0)$ *equals* $k-2$, *where* k *is the number of singularities of the fibration* f_H.

Example 4.3. In the case of $\mathrm{Sl}(2,\mathbb{C})$ the singular fibres are just the union of 2 subspaces. In this case the affine bundle has rank 0 and each fibre of this bundle intersects $H_0 + \mathfrak{n}^-(H_0)$ as well as $(w_0H_0) + \mathfrak{n}^-(w_0H_0)$ with $w_0H_0 = -H_0$. We conclude that this subbundle is contained in the affine spaces $H_0 + \mathfrak{n}^-(H_0)$ and $(w_0H_0) + \mathfrak{n}^-(w_0H_0)$ which are part of the singular levels of H_0 and $w_0H_0 = -H_0$, respectively.

REFERENCES

[Ab] Abouzaid, M. ; *Morse homology, tropical geometry, and homological mirror symmetry for toric varieties*. Selecta Math. (N.S.) **15** (2009), no. 2, 189–270.

[AAEKO] Abouzaid, M. ; Auroux, D. ; Efimov, A. ; Katzarkov, L. ; Orlov, D. ; *Homological mirror symmetry for punctured spheres*, J. Amer. Math. Soc. **26** (2013), 1051–1083 .

[AbS] Abouzaid, M. ; Smith, I. ; *Homological mirror symmetry for the 4-torus*. Duke Math. J. 1**52** (2010), no. 3, 373–440.

[ABKP] Amorós, J. ; Bogomolov, F. ; Katzarkov, L. ; Pantev, T. ; *Symplectic Lefschetz fibrations with arbitrary fundamental groups*, J. Differential Geom. **54** (2000), no. 3, 489–545.

[At] Atiyah,M.; Convexity and commuting hamiltonians, Bull. London Math. Soc. **14** (1982), 1–15.

[Au] Auroux, D. ; *A beginner's introduction to Fukaya categories*, Arxiv:1301.7056.

[AKO1] Auroux, D.; Katzarkov, L.; Orlov, D.; *Mirror symmetry for weighted projective planes and their noncommutative deformations*, Ann. Math. **167** (2008), 867–943.

[AKO2] Auroux, D. ; Katzarkov, L. ; Orlov, D. ; *Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves* Inventiones Math. **166** (2006), 537–582.

[ABB] Azad, H.; van den Ban, E.; Biswas, I.; *Symplectic geometry of semisimple orbits*. Indag. Mathem. , N. S. **19** (4) (2008), 507–533.

[Cl] Clarke, P. ; *Duality for Landau–Ginzburg models*, arXiv:0803.0447.

[Do] Donaldson, S. K .; *Lefschetz fibrations in symplectic geometry*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), Doc. Math. Extra Vol. II (1998), 309–314.

[DKV] Duistermaat, J. J.; Kolk J. A. C.; Varadarajan, V. S.; *Functions, flows and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups*. Compositio Math., **49** (1983), 309–398.

[E] Efimov, A. ; *Homological mirrror symmetry for curves of higher genus* Adv. Math. **230** (2012), no. 2, 493–530.

[EG] Eliashberg,Y. ; Gromov, M. ; *Convex symplectic manifolds*. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 135–162.

[FSS] Fukaya, K. ; Seidel, P. ; Smith, I. ; *Exact lagrangian submanifolds in simply-connected cotangent bundles*, Invent. Math, **172** (2008), 1–27.

[FOOO] Fukaya, K ; Oh, Y. ; Ohta, H. ; Ono, K. ; *Lagrangian intersection Floer theory: anomaly and obstruction*. Part I. AMS/IP Studies in Advanced Mathematics, 46.1. American Mathematical Society, Providence, RI; International Press, Somerville, MA (2009).

[F] Fukaya, K. ; *Mirror symmetry of abelian varieties and multi-theta functions*. J. Algebraic Geom. **11** (2002), no. 3, 393–512.

[GGS] Gasparim, E. ; Grama, L. ; San Martin, L. A. B. ; *Adjoint orbits of semisimple Lie groups and Lagrangian submanifolds*, arXiv:1401.2418.

[Go] Gompf, R. E.; *Symplectic structures from Lefschetz pencils in high dimensions*, Geometry & Topology Monographs **7**: Proceedings of the Casson Fest (2004) 267–290.

[GoS] Gompf, R. ; Stipsicz, A. ; *An introduction to 4-manifolds and Kirby calculus*, Graduate Studies in Mathematics **20**, American Math. Society, Providence (1999).

[GS] Guillemin, V. ; Sternberg, S. ; *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.

[GSh] Guffin, J. ; Sharpe, E. ; *A-twisted Landau-Ginzburg models*. J. Geom. Phys. **59** (2009), no. 12, 1547–1580.

[KP] Katzarkov, L. ; Przyjalkowski, V. ; *Landau–Ginzburg modelsÑold and new*. Proceedings of the GŽkova Geometry-Topology Conference 2011, 97–124, Int. Press, Somerville, MA (2012).

[Kc] Kocherlakota, R.R.; Integral Homology of real flag manifolds and loop spaces of symmetric spaces. Adv. Math **110** (1995), 1–46.

[Ko] Kontsevich, M. ; *Homological algebra of Mirror Symmetry*, Proc. International Congress of Mathematicians (Zurich, 1994) Birkhäuser, Basel (1995) 120–139.

[K] Kostant, B., On convexity, the Weyl group and the Iwasawa decomposition, Ann. Sci. Ecole Norm. Sup. **6** (1973), 413–455.

[KS] Khovanov, M.; Seidel, P. ; *Quivers, Floer cohomology, and braid group actions*, Journal: J. Amer. Math. Soc. **15** (2002), 203–271.

[PZ] Polishchuk, A.; Zaslow, E.; *Categorial mirror symmetry: The elliptic curve*, Adv. Theor. Math. Phys. **2** (1998) 443–470.

[SM] San Martin, L.A.B.; *Álgebras de Lie*, segunda edição, editora Unicamp (2010).

[Se] Seidel, P. ; *Vanishing cycles and mutations*, European Congress of Mathematics, Vol. II (Barcelona, 2000), 65–85, Progr. Math. **202**, Birkhäuser, Basel (2001).

[Se1] Seidel, P. ; *Homological mirror symmetry for the genus two curve*. J. Algebraic Geom. **20** (2011), no. 4, 727–769.

[Sh] Sheridan, N. ; *Homological Mirror Symmetry for Calabi–Yau hypersurfaces in projective space*, arXiv:1111.0632.

[S] Smith, I. ; *Floer cohomology and pencils of quadrics.* Invent. Math. **189** (2012), no. 1, 149–250.

IMECC - UNICAMP, DEPARTAMENTO DE MATEMÁTICA. RUA SÉRGIO BUARQUE DE HOLANDA, 651, CIDADE UNIVERSITÁRIA ZEFERINO VAZ. 13083-859 CAMPINAS - SP, BRASIL. E-MAILS: SMARTIN@IME.UNICAMP.BR, ETGASPARIM@GMAIL.COM, LINOGRAMA@GMAIL.COM.