MATH 541- Part I - summary

Material covered before the first exam. Write examples for each of the concepts defined below.

I. Point Set Topology

1. *Topological space.* A topological space consists of a pair (X, τ) where X is a set and $\tau \subset P(X)$ is a family of subsets of X satisfying the 3 properties below:
   i) $\emptyset \in \tau$ and $X \in \tau$,
   ii) $U_1, \cdots, U_n \in \tau$ implies $\cap_{i=1}^{n} U_i \in \tau$,
   iii) $U_i \in \tau \forall i \in I$ implies $\cup_{i \in I} U_i \in \tau$.

2. *Continuity.* Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f \colon X \to Y$ is called *continuous* if $A \in \tau_Y \Rightarrow f^{-1}(A) \in \tau_X$.

3. *Connectedness.* X is *disconnected* is there exist $A, B \in \tau_X$ such that $X = A \cup B$ and $A \cap B = \emptyset$. X is *connected* if it is not disconnected.

4. *Hausdorff.* X is *Hausdorff* if for every pair of distinct points $x,y \in X$ there exist open sets $U_x, U_y \in \tau$ such that $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$.

5. *Compactness.* X is *compact* if every open cover of X has a finite subcover.

Some important results

6. In $\mathbb{R}^n$ continuity with epsilons and deltas is equivalent to topological continuity.

7. In $\mathbb{R}^n$ the topologies defined by the ℓ_1, ℓ_2 and ℓ_∞ norms are equivalent.

8. If X is Compact and F is a closed subset of X, then F is compact.

9. If X is Hausdorff and K is a compact subset of X, then K is closed.

II. Algebraic Topology

10. *Homotopy of paths.* Let $\alpha: I \to X$ and $\beta: I \to X$ be paths in X with same end points, that is, $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Then α is *homotopic* to β relative $(0, 1)$ if there exits a continuous map $H: I \times I \to X$ such that

$i.H(s, 0) = \alpha(s)\ \forall s$
$ii.H(s, 1) = \beta(s)\ \forall s$
$iii.H(0, t) = \alpha(0) = \beta(0)\ \forall t$
$iv.H(1, t) = \alpha(1) = \beta(1)\ \forall t.$

11. *Fundamental Group.* A *loop* in X with base point x is a closed path α in X with end points x, that is, $\alpha(0) = \alpha(1) = x$. Homotopy equivalence relative $(0, 1)$ defines an equivalence relation on the sets of loops in X with base point x. This set of equivalence classes forms a group with the operation of *concatenation* of loops, defined by

$$\alpha\beta(s) = \begin{cases} \alpha(2s)\ if\ 1 \leq s \leq 1/2 \\ \beta(1 - 2s)\ if\ 1/2 \leq s \leq 1 \end{cases}.$$

This group is called the *fundamental* group of X at x denoted $\pi_1(X, x)$.

12. Important result:

If X is path-connected then for any $x_1, x_2 \in X$ $\pi_1(X, x_1) \simeq \pi_2(X, x_2)$, and in this case we omit the base point and denote simply $\pi_1(X)$.

13. Some examples:

$$\pi_1(S^1) = \mathbb{Z}$$
$$\pi_1(R^n) = 0,\ \forall n \geq 1.$$

14. *Homotopy equivalence of spaces.* X and Y are *homotopy equivalent* written $X \sim Y$ if there exist maps $f: X \to Y$ and $g: X \to Y$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

15. Some examples:

$$\mathbb{R}^n \sim \{0\}.$$
$$\mathbb{R}^n - \{0\} \sim S^{n-1}.$$

16. *Products.* $\pi_1(X \times Y) = \pi_1(X) \oplus \pi_1(Y)$.

III. Important concepts

Write the definitions of:

17. *Differentiable manifolds*

A *differentiable manifold* M of class C^r and dimension n is a topological space satisfying:

- M is *locally Euclidean*, that is, for every point $x \in M$ there exists an open neighborhood U of x and a homeomorphism $\varphi\colon U \to \varphi(U) \subset \mathbb{R}^n$, called a *local chart* around x
- for every pair of local charts (U, φ) and (V, ψ) of M having $U \cap V \neq \emptyset$ the composite

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V)}\colon \varphi(U \cap V) \to \psi(U \cap V)$$

  is of class C^r. This composite is called a *transition function.*

Note 1: It is usual to require in the definition that M have a *maximal atlas.* That is, one defines the notion of *compatible* charts and demands that all compatible charts belong to the collection of charts associated to M. This is convenient, as for instance, it follows that any restriction of a local chart to a subset is also part of the atlas. However, given a manifold as we defined above, with any *atlas* (= a collection of charts that cover M), an application of Zorn's lemma shows that there exists a unique maximal atlas associated to M.

Note 2: By *smooth* manifold we mean a manifold of class C^∞. In this course, all our manifolds will be smooth manifolds.

18. *Complex manifolds* are manifolds with a finer structure, where instead of local charts to $\mathbb{R}^n$ we require local charts to $\mathbb{C}^n$ and instead of differentiable transition functions we require holomorphic (=analytic) transitions functions. Explicitly,

A *complex manifold* M of (complex) dimension n is a topological space satisfying:

- M is *locally complex*, that is, for every point $x \in M$ there exists an open neighborhood U of x and a homeomorphism $\varphi\colon U \to \varphi(U) \subset \mathbb{C}^n$, called a *local chart* around x

- for every pair of local charts (U, φ) and (V, ψ) of M having $U \cap V \neq \emptyset$ the composite

$$\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} \colon \varphi(U \cap V) \to \psi(U \cap V)$$

  is holomorphic.

19. *Lie Groups* are manifolds having a group structure. That is, a smooth manifold M is called a *Lie group* if it is endowed with a group operation $\star$ such that the *multiplication* map $(a, b) \to a \star b$ and the inverse operation $x \mapsto x^{-1}$ are continuous with respect to the manifold structure.