MATH 541- Part II - summary

Material covered before the second exam. Write examples for each of the concepts defined below.

I. Algebraic Topology

1. *Covering spaces.* $p: E \to X$ is a *covering space* of X if every $x \in X$ has an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets S_i in E each of which is mapped homeomorphically onto U by p. Such U are called *evenly covered.*

2. *Application.* We proved the path lifting lemma and the covering homotopy lemma. As an consequence we have that: $p_*: \pi_1(E) \to \pi_1(X)$ is a monomorphism.

3. *Covering transformations.* For any covering space $p: E \to X$, the group of *covering transformations* is the group of all homeomorphisms ϕ of E which preserve the fibres, that is $p \circ \phi = p$.

4. *Theorem.* Given a covering space $p: (E, e_0) \to (X, x_0)$ with group of covering transformations G. If E is simply connected and locally pathwise connected, then g is canonically isomorphic to $\pi_1(X)$.

5. *Example.* $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$.

Higher Homotopy Groups

6. *Higher homotopy.* Consider the space X^I with the compact-open topology, and denote by Ω_{x_o} the subspace of X^I consisting of all loops at x_0 (with base point the constant loop at x_o). We define for $n \geq 2$

$$\pi_n(X, x_0) = \pi_{n-1}(\Omega_{x_0}).$$

7. *Alternative definition.* We showed that $\pi_n(X, x_0)$ can be interpreted as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$.

8. *Homotopy Sequence of a Fibration.* To a fibration $F \to E \to X$ there is associated a long exact sequence

$$\to \pi_n(F) \to \pi_n(E) \to \pi_n(X) \to \pi_{n-1}(F) \to \cdots.$$

9. *Important Example.* Using the *Hopf Fibration*

$$S^1 \to S^3 \to S^2$$

it follows that

$$\pi_3(S^2) = \mathbb{Z}.$$

II. Differential Topology

Review of Calculus:

10. *Inverse Function Theorem.* Let $U \subset \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^n$ a C^r map $r \geq 1$. If Df_p is *invertible*, then f is a C^r local diffeomorphism at p.

11. *Local Form of Submersions.* Let $U \subset \mathbb{R}^m$ be open and $f: U \to \mathbb{R}^n$ a C^r map $r \geq 1$. Let $p \in U$, $f(p) = 0$, and suppose that Df_p is *surjective.* Then there exists a local diffeomorphism ϕ of $\mathbb{R}^m$ at 0 such that $\phi(0) = p$ and

$$f\phi(x_1, \cdots, x_m) = (x_1, \cdots, x_n).$$

12. *Local Form of Immersions.* Let $U \subset \mathbb{R}^m$ be open and $f: U \to \mathbb{R}^n$ a C^r map $r \geq 1$. Let $q \in \mathbb{R}^n$, such that $0 \in f^{-1}(q)$, and suppose that Df_0 is *injective.* Then there exists a local diffeomorphism ψ of $\mathbb{R}^n$ at q such that $\psi(q) = 0$, and

$$\psi f(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0 \cdots, 0).$$

Important definitions.

13. *Regular values.* Let $f: M \to N$ be a C^1 map. We call $x \in M$ a *regular point* if f is submersive at x, otherwise we call x a *critical point.* A point $y \in N$ is called a *regular value* if for all $x \in f^{-1}(y)$, x is a regular point.

14. *Regular Value Theorem.* Let $f: M \to N$ be a C^r map, $r \geq 1$. If $y \in f(M)$ is a regular value, then $Y = f^{-1}(y)$ is a C^r submanifold of M. Moreover, $dim(Y) = dim(M) - dim(N)$.

15. Some examples:
    S^n is a C^∞ submanifold of $\mathbb{R}^{n+1}$
    $SL(n)$ if a C^∞ submanifold of $GL(n)$.

III. Classification of Surfaces

We used the following two consequences of the Seifert–Van Kampen theorem:

16. *Theorem.* Suppose $X = U \cup V$ where U and V are open in X and $U \cap V$ is pathwise connected. If $\pi_1(U \cap V) = 0$, then $\pi_1(X) = \pi_1(U) \star \pi_1(V)$.

17. *Theorem.* Suppose $X = U \cup V$ where U and V are open in X and $U \cap V$ is pathwise connected. If $\pi_1(V) = 0$, then $\pi_1(X) = \pi_1(U)/[\pi_1(U \cap V)]$, where $[\pi_1(U \cap V)]$ denotes the smallest normal subgroup of $\pi_1(U)$ containing $i_*(\pi_1(U \cap V))$.

18. *Classification of Surfaces.* Any connected compact surface is either homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes.