MATH 542 - Part 1- summary

We discussed Simplicial Homology, Singular Homology, and Celular Homology. These homology theories have the common property that they satisfy the Eilenberg–Steenrod axioms.

I. Eilenberg Steenrod axioms for a Homology Theory.

(This part of the summary is copied from Spanier, Algebraic Topology p. 199. Note that Spanier writes the axioms for a homology with integer coefficients. Other rings can be used as coefficients.)

A *homology theory* H and ∂ consists of

a. A covariant functor H from the category of topological pairs and maps to the category of graded Abelian groups and homomorphisms of degree 0 [that is, $H(X,A) = \{H_q(X,A)\}$]

b. A natural transformation ∂ of degree -1 from the functor H on (X,A) to the functor H on $(A,\emptyset)$ [that is, $\partial_q(X,A)\colon H_q(X,A) \to (A)$].

These satisfy the following axioms

1. **Homotopy Axiom**. If $f_o, f_1\colon (X,A) \to (Y,B)$ are homotopic, then

$$H(F_0) = H(f_1)\colon H(X,A) \to H(Y,B)$$

2. **Exactness Axiom** For any pair (X,A) with inclusion maps $i\colon A \subset X$ and $j\colon X \subset (X,A)$ there is an exact sequence

$$\cdots \xrightarrow{\partial_{q+1}(X,A)} H_q(A) \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(j)} H_q(X,A) \xrightarrow{\partial_{q+1}(X,A)} H_{q-1}(A) \xrightarrow{H_{q-1}(i)} \cdots$$

3. **Excision Axiom** For any pair (X,A), if U is an open subset of X such that $\overline{U} \subset$int A, then the excision maps $j\colon (X \setminus U, A \setminus U) \subset (X,A)$ induces an isomorphism

$$H(j)\colon H(X \setminus U, A \setminus U) \simeq H(X,A)$$

4. **Dimension Axiom** If P is a one-point space, then

$$H_q(P) \simeq \begin{cases} 0 & q \neq 0 \\ \mathbb{Z} & q = 0. \end{cases}$$

This is the last of the axioms. One of the most important consequences of the axioms for a homology theory is the Mayer Vietoris sequence.

5. **Mayer Vietoris** If $X = A \cup B$ with A and B open in X, then the Mayer Vietoris long exact sequence gives $H_*(X)$ in terms of $H_*(A)$ and $H_*(B)$. The following is exact

$$\cdots \to H_q(A \cap B) \to H_q(A) \oplus H_q(B) \to H_q(X) \to H_{q-1}(A \cap B) \to \cdots$$

6. **Homology of spheres** We showed that:

$$H_q(S^n) = \begin{cases} \mathbb{Z} & if \quad q = 0 \, or \, n \\ 0 & \quad otherwise \end{cases} .$$

II. Degrees of maps.

We discussed several definitions of degrees of maps, without showing that these definitions are compatible with each other. We will retake this problem next semester. I recall here the two main definitions and some of their basic properties.

First recall that if M is a compact and orientable topological manifold (without boundary) of dimension n, then we saw that the top homology of M is $H_n(M) = \mathbb{Z}$, and moreover, the manifold itself can be regarded as a generator for this homology. This generator is referred to as the *orientation class* of M and we denote it by $\mathbf{1} \in \mathbb{Z}$.

7. **Topological degree** Let M and N be compact connected oriented topological manifolds (without boundary) and let $f\colon M \to N$ be a continuous map. The *topological degree* of f is by definition the integer $\deg(f) = f_*(\mathbf{1}) \in H_*(N)$.

8. **Smooth degree** Let M and N be compact connected oriented smooth manifolds (without boundary) and let $f\colon M \to N$ be a smooth map (=at least C^1). For a regular point x of f we set

$$\mathrm{sg}(x) = \begin{cases} +1 & if \quad \det(\mathrm{Jac}(df(x)) > 0 \\ -1 & if \quad \det(\mathrm{Jac}(df(x)) < 0 \end{cases} .$$

Let $y \in N$ be a regular value for f (which we know exists by Sard's lemma). The *smooth degree* of f is by definition the integer $\deg(f) = \sum_{x \in f^{-1}(y)} \mathrm{sg}(x)$.

We accepted (without proving) that these two degrees coincide for smooth maps, and that is why we are allowed to denote both by the common symbol deg(f). An interesting exercise is to show that the smooth degree is well defined, that is, to show that it does not depend upon the choice of regular value.

We used degrees of maps between spheres to prove:

9. **Hairy Ball theorem** The sphere S^n admits a nowhere vanishing tangent vector field if and only if n is odd.

It is evident from the topological definition of degree, that homotopic maps have the same degree. The converse is not true in general, but it is true if the target is a sphere.

10. **Hopf degree theorem** Two maps of a compact, connected, oriented $n-$manifold M into S^n are homotopic if and only if they have the same degree.

For a discussion of the proof, consult Guillemin and Pollack page 146. We will come back to this result next semester.