MATH 542- Part II - summary

Material covered before the second exam.

I. De Rham Cohomology

1. The *exterior differentiation* d on forms is defined as follows, given a $k-$form $\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1,\cdots,i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ on $\mathbb{R}^n$ the differential $\mathrm{d}\omega$ of ω is the $k+1-$form

$$d\omega = \sum_{i_1<\cdots<i_k} \sum_{\alpha=1}^{n} \frac{\partial \omega_{i_1,\cdots,i_k}}{\partial x^\alpha} \mathrm{d}x^\alpha \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

As a consequence of the fact that exterior differentiation commutes with pull-backs, that is, $f^*\mathrm{d}\omega = \mathrm{d}f^*\omega$, we define exterior differentiation for forms over a manifold.

Very important property: $\mathrm{d}^2 = 0$.

2. *Closed and exact forms.* A differentiable form η over a smooth manifold M is called

- *closed* if $\mathrm{d}\eta = 0$,
- *exact* if it is the differential of a $k-1-$form β over M, that is, $\eta = \mathrm{d}\beta$.

From the very important property, it follows that every exact form is closed. Cohomology measures precisely the failure of the inverse to hold.

3. *Cocycles.* A *$k-$cocycle* over M is by definition a closed $k-$form over M. The set of all $k-$ cocycles in M is classically denoted by $Z^k(M)$.

4. *Coboundaries.* A *$k-$coboundary* over M is by definition an exact $k-$form over M. The set of $k-$ coboundaries on M is classically denoted by $B^k(M)$.

5. The *$k-th$ De Rham Cohomology* group of M is the quotient

$$H^k(M) := \{k - cocycles\}/\{k - coboundaries\} = Z^k(M)/B^k(M).$$

6. *Homotopy invariance of De Rham Cohomology*: If M and N are homotopy equivalent, then $H^k(M) = H^k(N)$ for all $k \geq 0$.

7. *Examples*:

$$H^k(pt) = \begin{cases} 0 & if \quad k \neq 0 \\ \mathbb{R} & if \quad k = 0. \end{cases}$$

8. *Poicaré Lemma*: If A is any open star-shaped domain in $\mathbb{R}^n$ then any closed form in A is exact. In other words, $H^k(A) = 0$ far any star-shaped domain A and any $k > 0$. In particular, $H^k(\mathbb{R}^n) = 0$ for any $k > 0$.

9. The classical example of a differential form in $\mathbb{R}^2 - \{0\}$ is given by

$$\mathrm{d}\theta = \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y.$$

As a consequence $H^1(\mathbb{R}^2 - \{0\}) \neq 0$, in fact, by homotopy invariance

$$H^k(\{\mathbb{R}^2 - \{0\}) = H^k(S^1).$$

10. *Stokes Theorem*:

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega.$$

11. The main theorems of Calculus are particular instances of Stokes theorem. Exercise: show that the following theorems are consequences of Stokes Theorem:

- Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a)$$

- Stokes Theorem:

$$\int_M < (\nabla \times F), n > \mathrm{d}A = \int_{\partial M} < F, T > \mathrm{d}s$$

- Divergence Theorem:

$$\int_M div F \mathrm{d}V = \int_{\partial M} < F, n > \mathrm{d}A$$

- Green's Theorem:

$$\int_{\partial M} P\mathrm{d}x + Q\mathrm{d}y = \int\int_M \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y.$$

12. *Poincaré duality.* For a compact orientable manifold M of dimension n,

$$H^i(M) = H_{n-i}(M).$$