COHOMOLOGY GAPS FOR REFLEXIVE SHEAVES ON THREEFOLDS

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ABSTRACT. We discuss cohomological invariants of framed reflexive sheaves on $(n \geq 3)$-folds and show that some numerically admissible values of cohomology do not occur.

1. INTRODUCTION

We first mention briefly some physics motivation. The paper [GKM] showed gaps on the values of topological charges for instantons on local surfaces. Using the Kobayashi–Hitchin correspondence (see [LT] for the general theory on KH) this result is equivalent to the existence of gaps on the value of the holomorphic Euler characteristic of framed sheaves over local surfaces ([GKM], lemma 6.5 and proposition 6.7). For $(n \geq 3)$-folds there is no analogue of the beautiful KH correspondence, simply because the equation defining anti-self-dual connections only makes sense in 4 dimensions. Nevertheless, sheaves on threefolds are very interesting for questions in string theory and appear in various physical theories that can be regarded as higher dimensional analogues of instanton counting, such as counting of dyons and BPS states. In further generality sheaves on threefolds occur as D-branes in string theory and cohomological values of such sheaves can be interpreted as some measure of energy accounted for within the brane charge. Thus, we expect that gaps on cohomological invariants of sheaves on threefolds shall have interesting physics interpretations.

In this paper we show existence of gaps on cohomology values for sheaves on $(n \geq 3)$-folds and study their filtrability properties (theorem 1). We also discuss the deformation functor (definition 3) of framed sheaves on threefolds and present a cohomological criterion for rigidity (theorem 2). We present several concrete examples focusing on ruled and doubly-ruled threefolds (sections 2 and 5).

2. DOUBLY-RULED THREEFOLDS

In this section we define doubly-ruled threefolds, which will illustrate our theorems, and we set up the terminology of splitting types.

Fix a smooth and connected projective curve C of genus $q \geq 0$ and a rank 2 vector bundle G on C. Set $S := \mathbb{P}(G)$. Let $u \colon S \to C$ denote the ruling of S. Fix a rank 2 vector bundle A on S and set $X := \mathbb{P}(A)$. Let $v \colon X \to S$ be a ruling and $D \subset X$ a section of v such that $\mathcal{O}_X(D)$ is ample. Hence, we have 2 rulings and a section:

$$(1) \qquad X \overset{D}{\underset{v}{\rightleftarrows}} S \xrightarrow{\ u\ } C$$

such that $L := \mathcal{O}_D(D)$ is an ample line bundle on D. By assumption $v|D \to S$ is an isomorphism and we will use it to see $u\colon D \to C$ as a ruling of D. Here

$$h^1(X, \mathcal{O}_X) = h^1(D, \mathcal{O}_D) = q.$$

Definition 1. Set $a := u^{-1}(p) \cdot L$ for any $p \in C$. The integer a is called the *fiber degree* of L.

Fix a rank r vector bundle E_0 on D. For any $p \in C$ let $a_{1,p} \geq \cdots \geq a_{r,p}$ be the splitting type of $E_0|u^{-1}(p)$.

Definition 2. Set $a_i := a_{i,p}$, where p is a general point of C. The integers $a_1 \geq \cdots \geq a_r$ are called the *generic splitting type* of E_0.

Thus we have

$$
\begin{array}{ccc}
\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r) & & E_0 \\
\downarrow & & \downarrow \\
u^{-1}(p) & \longrightarrow & D.
\end{array}
$$

Let $b_1 \geq \cdots \geq b_{r^2}$ denote the generic splitting type of $End(E_0)$. Notice that $b_{r^2} = a_r - a_1$, $b_1 = a_1 - a_r$, and $b_1 - b_{r^2} = 2(a_1 - a_r)$.

3. Deformations of framed bundles

Let X be a smooth and connected projective n-fold and D a divisor on X. Fix a vector bundle E on X and consider the exact sequence

$$(2) \qquad 0 \to End(E)(-D) \to End(E) \to End(E|D) \to 0.$$

The cohomology exact sequence of (2) induces a linear map

$$\rho\colon H^1(X, End(E)) \to H^1(D, End(E|D))$$

and a surjection

$$(3) \qquad \eta\colon H^1(X, End(E)(-D)) \to \mathrm{Ker}(\rho).$$

Definition 3. $\mathrm{Ker}(\rho)$ is the *tangent space to the deformation functor* Ψ_{E_0} of E with prescribed restriction $E_0 := E|D$ to E.

Let X be a doubly ruled threefold as in (1). Fix any vector bundle E_0 on D and use the isomorphism $v|D$ to identify E_0 with a vector bundle F on S. Set $\mathbf{E_0} := v^*(F)$. Then $\mathbf{E_0}$ is a vector bundle on X such that $\mathbf{E_0}|D \cong E_0$.

Definition 4. We will say that $\mathbf{E_0}$ is the *trivial extension* of E_0 to X.

We first give a couple of examples of the extremal cases: (i) when $\mathbf{E_0}$ does not have any non-trivial deformation with fixed restriction to D, and (ii) when the map η appearing in (3) is an isomorphism.

Example 1. Take $E_0 \cong \mathcal{O}_D^{\oplus r}$. Hence $\mathbf{E_0} \cong \mathcal{O}_{\mathbf{X}}^{\oplus \mathbf{r}}$. Kodaira's vanishing gives $h^1(X, End(E)(-D)) = 0$. Hence, a trivial vector bundle is rigid with respect to the deformations which preserve its restriction to D. More generally, for $i = 0, 1$ and for all integers $k > 0$ we have $h^i(D, End(E_0) \otimes L^{*\otimes k}) = 0$, again by Kodaira vanishing. Since $h^1(X, \mathcal{O}_X) = q$, a trivial vector bundle on X is rigid without any condition on its restriction to D if and only if $q = 0$.

Example 2. Consider the case when E_0 simple and L is effective. Since $h^0(D, L) > 0$ and $L \neq \mathcal{O}_D$, then $h^0(D, End(E_0) \otimes L^{*\otimes k}) = 0$ for all integers $k > 0$ and the corresponding maps η of (3) are isomorphisms.

Example 3. Assume that the fiber degree is related to the splitting type so that $a > a_1 - a_r$ (see definitions 1 and 2). Restrict $End(E_0)$ to a general fiber T of the ruling $u \colon D \to C$. Notice that $End(E_0) \otimes L^{*\otimes k}|T$ has splitting type $b_1 - ka \geq \cdots \geq b_r - ka$. Since for all $k > 0$ we have $b_1 = a_1 - a_r$, $b_1 - ka < 0$, then $h^0(D, End(E_0) \otimes L^{*\otimes k}) = 0$ for all integers $k > 0$, and the corresponding maps η of (3) are isomorphisms.

Of course the above examples are special situations, and in general there will be deformations of E_0 and the map η will fail to be an isomorphism. We can also regard this situation from the opposite viewpoint; instead of looking at $\mathbf{E_0}$ as an extension of E_0, we can regard $\mathbf{E_0}$ as a framed bundle, that is, $\mathbf{E_0}$ is given together with its restriction to a divisor. We then find that such framing imposes restrictions to the cohomology of $\mathbf{E_0}$.

4. COHOMOLOGY OF FRAMED BUNDLES

Proposition 1. *Let X be a smooth and connected projective n-fold, $n \geq 3$, and D an effective, smooth and ample divisor of X. Let E be a rank r vector bundle on X such that $E|D$ is trivial. Then E is trivial.*

Proof. Since $\det(E|D)$ is trivial and D is ample, $\det(E) \cong \mathcal{O}_X$. For any integer t consider the exact sequence

$$(4) \qquad 0 \to E(-(t+1)D) \to E(-tD) \to E(-tD)|D \to 0.$$

Since $\mathcal{O}_D(D)$ is ample, $E|D$ is trivial and $\dim(D) \geq 2$, we have $h^i(D, E(-tD)|D) = 0$ for $i = 0, 1$ and all $t \geq 1$ (Kodaira's vanishing). Since E has depth at least 2 at each point of X a theorem of Serre gives $h^i(X, E(-tD)) = 0$ for $i = 0, 1$ and $t \gg 0$. Hence, by decreasing induction on t and using (4) we get $h^i(X, E(-tD)) = 0$ for $i = 0, 1$, and all $t \geq 1$. Thus, sequence (4) gives

$$(5) \qquad h^0(X, E) = h^0(D, E|D).$$

Since $E|D$ is trivial, we get $h^0(X, E) = r$. Let

$$(6) \qquad e_E \colon H^0(X, E) \otimes \mathcal{O}_X \to E$$

denote the evaluation map. Since e_E is a map between vector bundles of the same rank, it drops rank either at no point of X, or at all points of X, or else along a non-empty hypersurface of X. The exact sequence (4) for $t = 1$ together with (5) show that the restriction map $H^0(X, E) \to H^0(D, E|D)$ is an isomorphism. Since $E|D$ is trivial, e_E does not drop rank at any point of D. The ampleness of D implies that D intersects every curve contained in X and hence every surface contained in X. Hence e_E drops rank at no point of X, and consequently is an isomorphism. $\qquad \square$

Remark 1. (choice of framing) Let X be a smooth and connected projective n-fold, $n \geq 3$, and D an effective and ample divisor of X. Fix $R \in \mathrm{Pic}(X)$ and a rank r vector bundle G on D. There is a natural bijection $E \mapsto E \otimes R$ between the rank r vector bundles (or reflexive sheaves) E on X such that $E|D \cong G$ and the rank r vector bundles (resp. reflexive sheaves) F on X such that $F|D \cong G \otimes (R|D)$.

Hence we see that we gain nothing if instead of assuming $E|D \cong \mathcal{O}_D^{\oplus r}$ we assume $E|D \cong (R|D)^{\oplus r}$. To get a more interesting set-up we must either weaken the assumption "D ample" or else take a different vector bundle on D instead of a trivial vector bundle as a framing.

Proposition 2. *Let X be a smooth and connected projective n-fold, $n \geq 3$, and D an effective, smooth and ample divisor of X. Let E be a rank $r \geq 2$ reflexive sheaf on X such that $E|D \cong \mathcal{O}_D^{\oplus r}$. Then $\det(E) \cong \mathcal{O}_X$, and $h^0(X, E) \leq r$ with equality $h^0(X, E) = r$ if and only if E is trivial.*

Proof. We follow similar lines as in the proof of Proposition 1. By assumption E is locally free in a neighborhood of D and $\det(E)$ is a reflexive rank 1 sheaf. Since X is smooth, $\det(E)$ is a line bundle ([H2], Proposition 1.9). Since D is ample and $\det(E)|D \cong \det(E|D) \cong \mathcal{O}_D$, we get $\det(E) \cong \mathcal{O}_X$. Hence, we have the exact sequence (4) in this case as well. To get the inequality $h^0(X, E) \leq r$ we just use that E is torsion free thus with depth > 0 at each point of X; which gives $h^0(X, E(-tD)) = 0$ for $t \gg 0$ by a cohomological property of depth (Serre). If $h^0(X, E) = r$, then the evaluation map $e_E \colon H^0(X, E) \otimes \mathcal{O}_X \to E$ is surjective. Given that $e_E|D$ is an isomorphism and D is ample, e_E is an isomorphism outside a codimension 2 algebraic subset of X. Since E is reflexive, e_E is an isomorphism ([H2], Proposition 1.6). □

Theorem 1. *Let X be a smooth and connected projective n-fold with $n \geq 3$, and D an effective, smooth and ample divisor of X. Let E be a reflexive sheaf on X of rank $r \geq 2$ such that $E|D \cong \mathcal{O}_D^{\oplus r}$ and $v := h^0(X, E) > 0$. We have:*

- *(gap on cohomology) $v \neq r - 1$.*
- *(filtrability) If $v \neq r$, then there exists a rank $r - v$ reflexive sheaf G on X such that $h^0(X, G) = 0$, $G|D \cong \mathcal{O}_D^{\oplus(r-v)}$, and E fits into an exact sequence:*

$$0 \to \mathcal{O}_X^{\oplus v} \to E \to G \to 0.$$

Proof. Proposition 2 gives $v \leq r$. Assume $v \leq r - 1$ and consider the evaluation map e_E of (6). In a neighborhood of D the cokernel $G := \mathrm{Coker}(e_E)$ is a trivial rank $r - v$ vector bundle. Since D is ample, we get that G is locally free outside finitely many points of X. Fix any $P \in X$ at which G is not locally free. The exact sequence

$$(7) \qquad\qquad 0 \to H^0(X, E) \otimes \mathcal{O}_X \to E \to G \to 0$$

shows that G has depth at least $\max\{\mathrm{deph}(G), \mathrm{depth}(\mathcal{O}_X) - 1\}$ at P ([E], Cor. 18.6). Hence G has depth at least 2 at each point of X. Let G' be the torsion part of G. Assume $G' \neq 0$. We saw that G' has finite support and depth at least 2. However, any non-zero sheaf supported at finitely many points has depth 0, a contradiction. Thus G is torsion free, and consequently it is reflexive ([H2], Proposition 1.3). □

Now we consider vector bundles E and B on X and D resp. with $E|_D \simeq B$ and we will give conditions on B (we stress, on B, not on E) which assure that E is, up to isomorphism, the only vector bundle on X with B as its restriction to D.

Theorem 2. *(cohomological criterion for rigidity) Let X be a smooth and connected projective n-fold, $n \geq 3$, and D an effective, smooth and ample divisor of X. Fix a rank r vector bundle B on D such that $h^i(D, Hom(B, B) \otimes \mathcal{O}_D(-tD)) = 0$ for*

$i = 0, 1$, and all integers $t \geq 1$. Then, up to isomorphisms, there is at most one vector bundle E on X such that $E|D \cong B$.

Proof. Let E, A be vector bundles on X such that $E|D \cong A|D \cong B$. For any integer x there is an exact sequence on X:

(8)
$$0 \to Hom(E, A)(-(x+1)D) \to Hom(E, A)(-xD) \to Hom(B, B) \otimes \mathcal{O}_D(-xD) \to 0.$$

Since $Hom(E, A)$ is locally free, X is smooth and $\dim(X) \geq 2$, a theorem of Serre gives $h^i(X, Hom(E, A)(-zD)) = 0$ if $z \gg 0$. Hence, from (8) our assumptions easily give $h^1(X, Hom(B, B)(-D)) = 0$. Thus (8) implies that the identity map $id_B : B \to B$ lifts to a homomorphism $u : E \to A$ which is an isomorphism at each point of D. Since X is connected and E, A are vector bundles on X with the same rank, either u is an isomorphism or it drops rank on an effective divisor T (the zero locus of the determinant of u). Since $u|D = id_B$ is an isomorphism, $T \cap D = \emptyset$. But given that D is ample, it meets every curve of X and hence every non-empty effective divisor of D, a contradiction. $\square$

Remark 2. (nonexistence of extensions) Of course, for many vector bundles B on D there is no vector bundle extension to X such that $E|D \cong B$ (take for instance the case $r = 2$, $X = \mathbb{P}^n$, $n \geq 3$, and D a hyperplane).

Example 4. Let X be a smooth and connected projective n-fold, $n \geq 3$, and D an effective, smooth and ample divisor of X. For any integer $x \geq 0$ let $D_X^{(x)}$ denote the infinitesimal neighborhood of order x of D in X, i.e. the closed subscheme of X with $\mathcal{O}_X(-(x+1)D)$ as its ideal sheaf. For instance $D_X^0 = D$. Fix any vector bundle B on D. Since D is ample and $\dim(D) = n - 1 \geq 2$, there exists an integer $t_0 > 0$ (depending only on D, B and X) such that $h^i(D, Hom(B, B) \otimes \mathcal{O}_D(-tD)) = 0$ for $i = 0, 1$ and all integers $t \geq t_0$. Let E, E' be vector bundles on X such that $E|D_X^{(t_0-1)} \cong E'|D_X^{(t_0-1)}$ and $E|D \cong E'|D \cong B$. We claim that $E \cong E'$. Indeed, the proof of Proposition 2 shows that we may lift any isomorphism $\tau \colon E|D_X^{(t_0-1)} \to E'|D_X^{(t_0-1)}$ to a morphism $u \colon E \to E'$. Since $u|D = \tau|D$ is an isomorphism, E and E' are vector bundles with the same rank and D is ample, u is an isomorphism (see the proof of Proposition 1).

Example 5. Take X, D, B as in Example 4. Here we give an example where we may take $t_0 = 1$ and hence B has at most one extension to X. Assume the existence of an increasing filtration $\{B_i\}_{0 \leq i \leq r}$ of B by subsheaves such that $B_0 = 0$, $B_r = B$, and each B_i/B_{i-1}, $1 \leq i \leq r$, is an algebraically trivial line bundle. Hence $Hom(B, B)$ has an increasing filtration $\{A_i\}_{0 \leq i \leq r^2}$ by subsheaves such that $A_0 = 0$, $A_{r^2} = Hom(B, B)$ and each A_i/A_{i-1}, $1 \leq i \leq r^2$, is an algebraically trivial line bundle. Since ampleness is a numerical property for line bundles by the Nakai criterion of ampleness (see [H1]), each $(A_i/A_{i-1})^* \otimes \mathcal{O}_D(D)$, $1 \leq i \leq r^2$, is an ample line bundle on D. Using the short exact sequences of sheaves associated to the filtration $\{A_i\}_{0 \leq i \leq r^2}$ together with Kodaira vanishing, we obtain $h^i(D, Hom(B, B) \otimes \mathcal{O}_D(-tD)) = 0$ for $i = 0, 1$ and all integers $t > 0$. Thus, we can apply theorem 2 to find a unique extension of B. If $q := h^1(D, \mathcal{O}_D) > 0$, then the set of all algebraically trivial line bundles on D is an Abelian variety of dimension q and for each integer $r \geq 2$ there exists an indecomposable rank r vector bundle B on D with a filtration as above. In the set-up of doubly ruled 3-folds q is the genus of the base curve C.

5. Further examples of bundles and sheaves on threefolds

In our general set-up we have $X = \mathbb{P}(A)$ where A is a rank 2 bundle on a surface S. Now we consider special cases where we have more concrete information.

Example 6. Fix an ample line bundle L on S and set $X' := \mathbb{P}(\mathcal{O}_X \oplus L)$. Let $v' : X' \to S$ denote the ruling. The projection $\pi_1 : \mathcal{O}_S \oplus L \to \mathcal{O}_S$ induces a section of v' whose image D' has normal bundle $\pi^*(L)|S_1$. Hence D has ample normal bundle and $X \backslash D \cong \mathrm{Tot}(L)$. However D' is not an ample divisor of X'.

The projection $\mathcal{O}_S \oplus L \to L$ induces another section D'' of v' whose normal bundle $\pi_2 : \mathcal{O}_{D''}(D'')$ is isomorphic to L^*(up to the identification of D'' with S given by the ruling v'). Hence the conormal bundle of D'' in X' is ample. Since the kernels of the projections π_1 and π_2 are complementary, $D' \cap D'' = \emptyset$. We may see D'' as the obstruction to the ampleness of D' in X'. For all integers $r > 0$, $c_1, \ldots, c_r$, set

$$A\{c_1, \ldots, c_r\} := \oplus_{i=1}^{r} \mathcal{O}_{X'}(c_i D'').$$

Since $D'' \cap D' = \emptyset$, we have $A\{c_1, \ldots, c_r\}|_{D'} \cong \mathcal{O}_{D'}^{\oplus r}$. Restricting to each D'' we see that $A\{c_1, \ldots, c_r\} \cong A\{d_1, \ldots, d_r\}$ if and only if there is a permutation $\sigma : \{1, \ldots, r\} \to \{1, \ldots, r\}$ such that $d_i = c_{\sigma(i)}$ for all i. Obviously, $h^0(X', \mathcal{O}_{X'}(cD'')) = 0$ if $c < 0$. Since L is ample, we see that D'' appears with multiplicity c in the base locus of the linear system $|cD''|$ on X'. Hence

$$h^0(X, A\{c_1, \ldots, c_r\}) = \#\{i \in \{1, \ldots, r\} : c_i \geq 0\}.$$

Example 7. Let $\alpha : U \to V$ be a $\mathbb{P}^1$-bundle. Hence every scheme-theoretic geometric fiber F of α is isomorphic to $\mathbb{P}^1$. Thus $h^0(F, \mathcal{O}_F) = 1$ and $h^i(F, \mathcal{O}_F) = 0$ for all $i \geq 1$. The first equality gives $\alpha_*(\mathcal{O}_U) = \mathcal{O}_V$. Since $h^i(F, \mathcal{O}_F) = 0$ for all $i \geq 1$, a base-change theorem ([OSS], p. 11) gives $R^i\alpha_*(\mathcal{O}_U) = 0$ for all $i \geq 1$. Now assume that V is complete. Hence U is complete. The Leray spectral sequence of α gives $h^i(U, \mathcal{O}_U) = h^i(V, \mathcal{O}_V)$ for all $i \geq 0$. Applying the observations to the rulings $X \to D$ and $X' \to D'$ we get

$$h^1(X, \mathcal{O}_X) = h^1(X', \mathcal{O}_{X'}) = q$$

and

$$h^i(X, \mathcal{O}_X) = h^i(X', \mathcal{O}_{X'}) = 0$$

for all $i \geq 2$.

Proposition 3. *With D' and D'' as in example 6 we have:*

- $h^1(X, \mathcal{O}_{X'}(-D')) = h^1(X, \mathcal{O}_{X'}(-D'')) = 0$.
- $h^1(X', \mathcal{O}_{X'}(xD'' - D')) = 0$ *for all integers $x \geq 0$.*
- *Fix an integer $y > 0$, then*

$$h^1(X', \mathcal{O}_{X'}(-yD'' - D')) \leq \sum_{i=1}^{} yh^0(S, L^{\otimes i}).$$

- *If $h^1(S, L^{\otimes i}) = 0$ for all $1 \leq i \leq y$, then*

$$h^1(X', \mathcal{O}_{X'}(-yD'' - D')) = \sum_{i=1}^{} yh^0(S, L^{\otimes i}).$$

Proof. Consider the exact sequences on X'

$$(9) \qquad 0 \to \mathcal{O}_{X'}(cD'' - D') \to \mathcal{O}_{X'}(cD'' - D') \to \mathcal{O}_{D'}(cD'') \to 0$$

(10) $0 \to \mathcal{O}_{X'}((x-1)D'' - D') \to \mathcal{O}_{X'}(xD'' - D') \to \mathcal{O}_{D''}(xD'' - D') \to 0$

Recall that $\mathcal{O}_{D'}(cD'') \cong \mathcal{O}_{D'}$ for any c because $D' \cap D'' = \emptyset$, also $\mathcal{O}_{D''}(xD'') \cong L^{\otimes -x}$. Hence $\mathcal{O}_{D''}(xD'' - D') \cong L^{\otimes -x}$. First take $c = 0$ in sequence (9), then the natural map $H^1(X, \mathcal{O}_{D'}) \to H^1(D, \mathcal{O}_{D'})$ is an isomorphism (use that $X' \to D'$ is a $\mathbb{P}^1$-bundle and the proof that $h^1(X, \mathcal{O}_{X'}) = q$ given in Example 7). The restriction map $H^0(X, \mathcal{O}_{X'}) \to H^0(X, \mathcal{O}_{X'})$ is obviously bijective. Hence $h^1(X, \mathcal{O}_X(-D)) = 0$. In the same way using that $X' \to D''$ is a $\mathbb{P}^1$-bundle we get $h^1(X, \mathcal{O}_{X'}(-D'')) = 0$.

Now we check that $h^1(X', \mathcal{O}_{X'}(xD'' - D')) = 0$ by induction on x. We proved that this vanishing is true if $x = 0$. Assume $x > 0$ and that this vanishing is true for the integer $x' := x - 1$. Since L is ample on D'' and $x > 0$, Kodaira vanishing gives $h^1(X, L^{\otimes -x}) = 0$. Hence the sequence (10) gives $h^1(X', \mathcal{O}_{X'}(xD'' - D')) = 0$.

We check the last two assertions by induction on y, starting from the case $y = 0$ which is true, because $h^1(X, \mathcal{O}_{X'}(-D')) = 0$. Assume that the last assertions are true for the integer $y' := y - 1$. Then use the sequence (10) with $x := y - 1$ to prove both assertions. $\qquad\square$

Proposition 4. $h^2(S, L^{\otimes k}) = 0$ for all $k \geq 0$.

Proof. Let $u \colon S \to C$ be the ruling of S. By Serre duality it is sufficient to prove $h^0(S, L^{\otimes -k} \otimes \omega_S) = 0$, i.e. that $h^0(C, u_*(L^{\otimes -k} \otimes \omega_S)) = 0$. Let $T \cong \mathbb{P}^1$ be a fiber of u. Since L is ample, $L|T$ has degree > 0. Since ω_S has degree -2 and $k \geq 0$, $L^{\otimes -k} \otimes \omega_S|T$ has degree < 0. Hence $u_*(L^{\otimes -k} \otimes \omega_S) = 0$ and it follows that $h^0(C, u_*(L^{\otimes -k} \otimes \omega_S)) = 0$. $\qquad\square$

Example 8. Fix an integer $k > 0$ and assume $h^1(X, L^{\otimes k}) = 0$. Proposition 4 gives $h^2(S, L^{\otimes k}) = 0$. By Riemann–Roch $h^0(S, L^{\otimes k}) = k^2 L^2 - k\omega_S \cdot L + 1 - q$. Consider the following property that L may have:

   property (+) $h^1(S, L^{\otimes i}) = 0$ for all integers $i > 0$.

If L has Property (+), then Proposition 3 shows that the integer

$$h^1(X, Hom(A_{c_1,\dots,c_r}, A_{c_1,\dots,c_r})(-D'))$$

is uniquely determined by the integers $r, c_1, \dots, c_r$ and the intersection numbers L^2 and $\omega_S \cdot L$. Recall that $H^1(X, Hom(A\{c_1, \dots, c_r\}, A\{c_1, \dots, c_r\})(-D'))$ is the tangent space at $A\{c_1, \dots, c_r\}$ of the deformation functor of $A\{c_1, \dots, c_r\}$ with fixed restriction to D'.

Fix any ample line bundle G on S. Since G is ample, there is an integer $k_G > 0$ such that $h^1(S, G^{\otimes k}) = 0$ for all integers $k \geq k_G$. Hence for any integer $k \geq k_G$ the ample line bundle $L := G^{\otimes k}$ has property (+). Thus, in all these cases the tangent space to the deformation functor is entirely determined by the numerical data $r, c_1, \cdots, c_r, L^2$, and $\omega_S \cdot L$.

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