The Atiyah–Jones conjecture for rational surfaces

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We show that if the Atiyah–Jones conjecture holds for a surface X, then it also holds for the blow-up of X at a point. Since the conjecture is known to hold for $\mathbb{P}^2$ and for ruled surfaces, it follows that the conjecture is true for all rational surfaces.

If $P \to X$ is a principal $SU(2)$ bundle over a Riemannian four-manifold X, with $c_2(P) = k > 0$, and A is a connection on P, the Yang-Mills functional

$$YM(A) \;=\; \int_X ||F_A||^2$$

is minimal precisely when the curvature F_A is anti-self dual, i.e. $F_A = -*F_A$, in which case A is called an instanton of charge k on X.

Let $\mathcal{M}I_k(X)$ denote the moduli space of framed instantons on X with charge k and let $\mathcal{C}_k(X)$ denote the space of all framed gauge equivalence classes of connections on X with charge k. In 1978, Atiyah and Jones [AJ] conjectured that the inclusion $\mathcal{M}I_k(X) \to \mathcal{C}_k(X)$ induces an isomorphism in homology and homotopy through a range that grows with k. The original statement of the conjecture was for the case when X is a sphere, but the question readily generalises for other 4-manifolds.

The stable topology of these moduli spaces was understood in 1984, when Taubes [Ta] constructed instanton patching maps $t_k\colon \mathcal{M}I_k(X) \to \mathcal{M}I_{k+1}(X)$ and showed that the stable limit $\lim\limits_{k\to\infty} \mathcal{M}I_k$ indeed has the homotopy type of $\mathcal{C}_k(X)$. However, understanding the behaviour of the maps t_k at finite stages is a finer question. Using Taubes' results, to prove the Atiyah–Jones conjecture it then suffices to show that the maps t_k induce isomorphism in homology and homotopy through a range.

In 1993, Boyer, Hurtubise, Milgram and Mann [BHMM] proved that the Atiyah–Jones conjecture holds for the sphere S^4 and in 1995, Hurtubise and Mann [HM] proved that the conjecture is true for ruled surfaces.

In this paper I show that if the Atiyah–Jones conjecture holds true for a complex surface X then it also holds for the surface $\widetilde{X}$ obtained by blowing-up X at a point. In particular, it follows that the conjecture holds true for all rational surfaces.

Kobayashi and Hitchin gave a one-to-one correspondence between instantons on a topological bundle E over X and holomorphic structures on E, see [LT]. Using this correspondence I translate the Atiyah–Jones conjecture into the language of holomorphic bundles and compare the topologies of the moduli spaces $\mathfrak{M}_k(X)$ and $\mathfrak{M}_k(\widetilde{X})$ of stable holomorphic bundles on X, resp. $\widetilde{X}$, having $c_1 = 0$ and $c_2 = k$. More precisely, I will consider slightly enlarged moduli spaces $\mathfrak{M}_k^f(\widetilde{X})$ of bundles on $\widetilde{X}$ framed near, but not *on*, the exceptional divisor ℓ. That is, we fix a small tubular neighborhood $N(\ell)$ of ℓ in $\widetilde{X}$ and assign frames on $N^0 := N(\ell) - \ell$. Existence of such frames is guarantied by [G2,Thm. 4.1]. These framed moduli spaces $\mathfrak{M}_k^f(\widetilde{X})$ that are naturally related to the moduli spaces $\mathfrak{M}_k^f(X)$ by a holomorphic gluing construction (Proposition 4.1).

The structure of the proof is the following. First we give a concrete description of instantons on $\widetilde{\mathbb{C}^2}$ (section 1) and compute its numerical invariants (section 2). We then study moduli spaces $\mathfrak{M}_k(\widetilde{X})$ of instantons or equivalently stable bundles on $\widetilde{X}$, with respect to a polarisation $\widetilde{\mathcal{L}} = N\mathcal{L} - \ell$ where $\mathcal{L}$ is a polarisation on X. We show that removing the singular points of this moduli space does not affect homology in a range sufficient to our calculations and thereafter work only with its smooth points. Moreover, the direct image of a stable bundle might yield an unstable bundle, which does not correspond to an instanton on X. We show that removing such unstable bundles from $\mathfrak{M}_k(X)$ does not affect our homology calculations either (section 3).

We show that any framed instanton on $\widetilde{X}$ is uniquely determined by holomorphic patching of a framed instanton on X and an instanton on $\widetilde{\mathbb{C}^2}$ (section 4). We define framed moduli spaces (section 5) and then prove that the local moduli space $\mathcal{N}_i^f$ of (framed) instantons on $\widetilde{\mathbb{C}^2}$ with charge i is a smooth complex manifold (section 6). As a consequence we have stratifications

$$\mathfrak{M}_k^f(\widetilde{X}) \simeq \bigcup_{i=0}^{k} \mathfrak{M}_{k-i}^f(X) \times \mathcal{N}_i^f.$$

We prove that the map $\mathfrak{M}_k^f(\widetilde{X}) \to \mathfrak{M}_{k+1}^f(\widetilde{X})$ obtained by translating Taubes' map to bundles via Kobayashi–Hitchin correspondence is homotopic equivalent to a map that preserves the stratifications (section 7). Using Leray spectral sequences, we then show that Atiyah–Jones conjecture for $\widetilde{X}$ follows from the corresponding statement for X.

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1. Moduli of bundles on $\widetilde{\mathbb{C}^2}$ with fixed splitting type

Let $\pi\colon \widetilde{X} \to X$ be the blow-up of a point x on a compact surface X and let ℓ be the exceptional divisor. In this paper all bundles have rank 2 and $c_1 = 0$. Given a bundle E over X we pull it back to $\widetilde{X}$ and then modify it by gluing in new data near the exceptional divisor to construct a bundle $\widetilde{E}$ satisfying $\widetilde{E}|_{\widetilde{X}-\ell} \simeq E|_{X-x}$. The difference of Chern classes $c_2(\widetilde{E}) - c_2(E)$ depends only on the restriction of $\widetilde{E}$ to a small neighborhood $N(\ell)$ of ℓ. To begin with, suppose that $N(\ell)$ is isomorphic to the blow-up of $\mathbb{C}^2$ at the origin, denoted $\widetilde{\mathbb{C}^2}$. Given a rank two bundle V on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class, there exists an integer $j \geq 0$ determined by the restriction of V to the exceptional divisor, such that $V|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(j) \oplus \mathcal{O}_{\mathbb{P}^1}(-j)$. This j is called the *splitting type* of the bundle. We denote by $\mathcal{O}_{\widetilde{\mathbb{C}^2}}(j)$ the unique line bundle on $\widetilde{\mathbb{C}^2}$ having first Chern class j, that is, the pull back of $\mathcal{O}_{\mathbb{P}^1}(j)$. We choose *canonical* coordinates for $N(\ell) = U \cup V$, where $U \simeq \mathbb{C}^2$ has coordinates (z, u) and $V \simeq \mathbb{C}^2$ has coordinates (ξ, v) with $(\xi, v) = (z^{-1}, zu)$ in $U \cap V \simeq (\mathbb{C} - 0) \times \mathbb{C}$.

Theorem [G3] *Every holomorphic rank two bundle V over $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class and splitting type j is an algebraic extension of the form*

$$0 \to \mathcal{O}_{\widetilde{\mathbb{C}^2}}(-j) \to V \to \mathcal{O}_{\widetilde{\mathbb{C}^2}}(j) \to 0$$

and the extension class can be represented in canonical coordinates by a polynomial of the form

$$p = \sum_{i=1}^{2j-2} \sum_{l=i-j+1}^{j-1} p_{il} z^l u^i.$$

It follows that the local (= over $N(\ell)$) moduli problem can be studied by considering extensions of line bundles modulo bundle isomorphism, even though this is definitely not the case for bundles on the compact surface $\widetilde{X}$. The fact that V is determined by an extension of degree $2j-2$ implies that V is determined over $N(\ell)$ by its restriction to the $2j-2$nd formal neighborhood of the exceptional divisor. Hence, our assumption that $N(\ell) \simeq \widetilde{\mathbb{C}^2}$ is not restrictive. Consider the quotient space

$$\mathcal{M}_j\colon = \{\textit{bundles on } \widetilde{\mathbb{C}^2} \textit{ with splitting type } j\}/\sim,$$

where $\sim$ denotes bundle isomorphism. $\mathcal{M}_j$ is topologized as follows: In canonical coordinates, the extension class is expressed by the complex polynomial p with $n = 2j(j+1)$ coefficients as in Theorem [G3]. In such canonical coordinates, elements of $\mathcal{M}_j$ are represented by transition matrices of the form

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}. \tag{1}$$

We set $p \sim p'$ if the corresponding bundles are holomorphically equivalent. This construction identifies $\mathcal{M}_j$ to a quotient of $\mathbb{C}^n$ under a group action and endows $\mathcal{M}_j$ with the structure of a quotient stack. For more on the structure of $\mathcal{M}_j$ see [G3] and [G5]. The stack $\mathcal{M}_j$ has a corse moduli space that can be decomposed into Hausdorff strata using the instanton numerical invariants, see [BG2]. To study the moduli problem we calculate numerical invariants.

2. Computation of local numerical invariants

In §1 we saw that an element $V \in \mathcal{M}_j$ is determined by its splitting type j and extension class p. We write $V = V(j,p)$. It is important to note that the bundle V is trivializable on the complement of ℓ.

LEMMA [G2] *Every holomorphic bundle on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class is trivial on $\widetilde{\mathbb{C}^2}$ minus the exceptional divisor.*

It follows that the local bundle $V(j,p)$ can be glued to any bundle E pulled back from X to form a new bundle $\widetilde{E}$ on $\widetilde{X}$. We also note that the topology of $\widetilde{E}$ does not depend on the attaching map. In fact, the attaching is given by a holomorphic map $\psi\colon \mathbb{C}^2 - \{0\} \to Sl(2,\mathbb{C})$. By Hartog's theorem ψ extends to the origin, and therefore is homotopic to the identity; thus not contributing to the Chern numbers of $\widetilde{E}$. Hence 2 sets of data (E,j,p,ψ) and (E,j,p,ψ') determine the same topological bundle, although in general they determine distinct holomorphic bundles.

LEMMA [G4] *Every holomorphic rank two bundle $\widetilde{E}$ on $\widetilde{X}$ with $c_1(\widetilde{E}) = 0$ is topologically determined by a triple (E,j,p) where E is a bundle over X, j is a non-negative integer and p is a polynomial.*

Here $E = \pi_*(\widetilde{E})^{\vee\vee}$ and the Chern class difference $c = c_2(\widetilde{E}) - c_2(E)$ depends only on the local data $V(j,p)$.

DEFINITION 2.1 We call c the *(local) charge* of the bundle $\widetilde{E}$ near the exceptional divisor. When considering the local situation, we also refer to c as the *(local) charge* of V.

The terminology comes from instanton moduli spaces, where c represents the change in topological charge of the instanton on $\widetilde{X}$ given by the patching of $V(j,p)$. Friedman and Morgan [FM] gave the bounds $j \leq c_2(\widetilde{E}) - c_2(E) \leq j^2$, and we proved sharpness.

THEOREM [G1] *The bounds* $j \leq c_2(\widetilde{E}) - c_2(E) \leq j^2$ *are sharp.*

In [BG1] we used elementary transformations to show that all intermediate values occur. The following result implies non-emptiness of the strata appearing in the stratification in §7.

THEOREM [BG1] *For every integer* k *satisfying* $j \leq k \leq j^2$ *there is a (semistable) holomorphic bundle on* $\widetilde{X}$ *with splitting type* j *and such that* $c_2(\widetilde{E}) - c_2(E) = k$.

In section 7 we use two finer numerical invariants, defined as follows. An application of Riemann–Roch ([FM], p. 366) gives

$$c = c_2(\widetilde{E}) - c_2(E) = l(R^1\pi_*\widetilde{E}) + l(Q), \tag{2}$$

where Q is the skyscraper sheaf defined by the exact sequence

$$0 \to \pi_*(\widetilde{E}) \to \pi_*(\widetilde{E})^{\vee\vee} \to Q \to 0,$$

and l denotes length. The pair of local analytic invariants $l(Q)$ and $l(R^1\pi_*\widetilde{E})$ gives strictly finer information than the local charge c, and has interesting properties such as:

THEOREM [BG2] *The pair of numerical invariants* $l(Q)$ *and* $l(R^1\pi_*\widetilde{E})$ *gives the coarsest stratification of* $\mathcal{M}_j$ *into maximal Hausdorff components.*

Sharp bounds and nonemptiness of intermediate strata are given by the following results.

THEOREM[G2] *Let* $j > 0$ *be the splitting type of* E, *then the following bounds are sharp*

$$j - 1 \leq l(R^1\pi_*(E)) \leq j(j-1)/2$$

and

$$1 \leq l(Q) \leq j(j+1)/2.$$

The upper bounds occur only at the split bundle. If $j = 0$ then both invariants are zero.

THEOREM[BG1]*For every pair of integers (w, h) satisfying $j - 1 \leq h \leq j(j-1)/2$ and $1 \leq w \leq j(j+1)/2$ with $j \geq 0$ there exists a rank 2 vector bundle E on $\widetilde{\mathbb{C}^2}$ with splitting type j having numerical invariants $l(R^1\pi_*(E)) = h$ and $l(Q) = w$.*

Non-emptiness of intermediate strata is needed for the proof of Lemma 7.3. The remainder of this section gives some examples of how these invariants are calculated. Other examples of calculations of these invariants appear in [G1] and [BG2]. In [GS] a Macaulay2 program that calculates both invariants is given, and the following simple formula for $l(R^1\pi_*\widetilde{E})$ is proven.

THEOREM [GS] *Let m denote the largest power of u dividing p, and suppose $m > 0$. If $\widetilde{E}$ is the bundle defined by data (E, j, p, ϕ), then*

$$l(R^1\pi_*\widetilde{E}) = \binom{j}{2} - \binom{j-m}{2}.$$

The following example illustrates a computation of $l(Q)$.

EXAMPLE 2.2 Let E be given by data $(j, z^n u); n \geq 1$. We show that $l(Q) = \binom{n}{2}$, it then follows from theorem [GS] and equality (2) that the charge is

$$c(E) = l(Q) + l(R^1\pi_*\widetilde{E}) = \binom{n}{2} + \binom{j}{2} - \binom{j-1}{2} = \binom{n}{2} + j.$$

We use the method of [BG2] section 6.2. Let $M = (\pi_*E)_0^\wedge$ denote the completion of the stalk $(\pi_*E)_0$. Let ρ denote the natural inclusion of M into its double dual $\rho : M \hookrightarrow M^{\vee\vee}$. We want to compute $l(Q) = dim\, coker(\rho)$. By the theorem on formal functions

$$M \simeq \lim_{\longleftarrow} H^0(\ell_n, \widetilde{V}|\ell_n),$$

where ℓ_n denotes the $n-$th infinitesimal neighborhood of ℓ. There are simplifications that make it easy to calculate M, (cf. [BG2] or [G1]). So, to determine M it suffices to calculate $H^0(\ell_{2j-2}, \widetilde{V}|\ell_{2j-2})$, and the relations among its generators under the action of $\mathcal{O}_0^\wedge(\simeq \mathbf{C}[[x, y]])$. In this example, E is given by transition matrix $\begin{pmatrix} z^j & z^n u \\ 0 & z^j \end{pmatrix}$. We find that $M = \mathbf{C}[[x,y]]\langle \beta_0, \beta_1, \cdots, \beta_n, \gamma\rangle$ where, for $1 \leq i \leq n$,

$$\beta_i = \begin{pmatrix} -z^i u \\ z^{j-n+i} \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 \\ u^{n-1} \end{pmatrix}$$

with relations

$$x\beta_i - y\beta_{i-1} = 0.$$

Consequently, $M^\vee = \langle B, C\rangle$ is free on the generators

$$B : \begin{cases} \beta_i \to x^i y^{n-i} \\ \gamma \to 0 \end{cases} \quad C : \begin{cases} \beta_i \to 0 \\ \gamma \to 1 \end{cases}.$$

$M^{\vee\vee} = \langle \mathcal{B}, \mathcal{C}\rangle$ is free on the generators

$$\mathcal{B} : \begin{cases} B \to 1 \\ C \to 0 \end{cases} \quad \mathcal{C} : \begin{cases} B \to 0 \\ C \to 1 \end{cases}.$$

The inclusion into the bidual $\rho : M \to M^{\vee\vee}$, takes x into evaluation at x, therefore

$$\rho : \begin{cases} \beta_i \to x^i y^{n-i}\mathcal{B} \\ \gamma \to \mathcal{C} \end{cases}$$

and $coker(\rho) = \langle x^i y^{n-i-1}\mathcal{B}; 0 \leq i \leq n\rangle$. We conclude that

$$l(Q) = dim\, coker(\rho) = \binom{n}{2}.$$

3. Moduli of bundles on $\widetilde{X}$

For the moduli problem on $\widetilde{X}$ we need stability conditions. If $\mathcal{L}$ is an ample divisor on X then, for large N, the divisor $\widetilde{\mathcal{L}} = N\mathcal{L} - \ell$ is ample on $\widetilde{X}$. We fix, once and for all, polarisations $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ on X and $\widetilde{X}$ respectively. For a polarised surface Y, the notation $\mathfrak{M}_k(Y)$ stands for moduli of rank 2 bundles on Y having $c_1 = 0$, $c_2 = k$ and slope stable with respect to the fixed polarisation. If E is $\mathcal{L}$-stable on X, then $\pi^*(E)$ is $\widetilde{\mathcal{L}}$-stable on $\widetilde{X}$ [FM, Thm. 6.5]. Hence, the pull back map induces an inclusion of moduli spaces $\mathfrak{M}_k(X) \hookrightarrow \mathfrak{M}_k(\widetilde{X})$.

Our objective is to show that given a map $\mathfrak{M}_k(X) \to \mathfrak{M}_{k+1}(X)$ inducing a homology equivalence through a range, there is a map $\mathfrak{M}_k(\widetilde{X}) \to \mathfrak{M}_{k+1}(\widetilde{X})$ inducing a homology equivalence through a comparable range, and that both ranges go to ∞ with k. Given that the homology equivalence for the case of $X = \mathbb{P}^2$ holds up to dimension $\lfloor k/2 \rfloor - 1$, we expect to obtain an equivalence range at best equal to this one. From here on, we are free to modify $\mathfrak{M}_k(X)$ and $\mathfrak{M}_k(\widetilde{X})$ in any way that does not change topology up to dimension $k/2 = c_2/2$. Firstly, we remove singularities of the moduli spaces; secondly, we remove semistable bundles on $\mathfrak{M}_i(X)$ that are not stable.

3.1 Removing singularities

We remove the singular points of the moduli spaces, in order to work only with smooth manifolds. We use the following results of Kirwan and Donaldson.

THEOREM ([Ki], Cor. 5.4) *Let X be a quasi-projective variety and m a non-negative integer such that every $x_0 \in X$ has a neighborhood in X isomorphic to*

$$\{x \in \mathbb{C}^N | f_1(x) = \cdots = f_M(x) = 0\}$$

for some integers N, M, and holomorphic functions f_i depending on x_0 with $M \leq m$. If Y is a closed subvariety of codimension k in X, then for $q < k - m$,

$$H_q(X - Y, \mathbb{Z}) \simeq H_q(X, \mathbb{Z}).$$

Given a complex surface S with polarisation H, let $\Sigma_k \subset \mathfrak{M}_k(S)$ denote the algebraic subvariety representing bundles E with $H^2(\mathrm{End}_0 E) \neq 0$. A local model of the moduli space is determined by the kernel of the Kuranishi map $Sym^2 H^1(\mathrm{End}_0 E) \to H^2(\mathrm{End}_0 E)$ parameterising small deformations of E. In case $H^2(\mathrm{End}_0 E) = 0$, small deformations of E are unobstructed and E is a smooth point of the moduli space. Therefore, singular points satisfy $H^2(\mathrm{End}_0 E) \neq 0$ and the singular part of $\mathfrak{M}_k$ is contained in Σ_k.

THEOREM ([Do], Thm. 5.8) *There are constants a, b depending only on S and the ray spanned by H in $H_2(S)$ such that:*

$$\dim_{\mathbb{C}} \Sigma_k \leq a + b\sqrt{k} + 3k.$$

PROPOSITION 3.1 *Removing the singularity set $Sing$ of $\mathfrak{M}_k(S)$ does not change homology in dimension less than k. That is, for $q < k$*

$$H_q(\mathfrak{M}_k(S)) = H_q(\mathfrak{M}_k(S) - Sing).$$

Proof. By Kuranishi theory, points $E \in \mathfrak{M}_k(S)$ satisfying $H^2(\mathrm{End}_0 E) = 0$ are smooth points. Therefore, the singularity set of $\mathfrak{M}_k(S)$ is contained in Σ_k.

Moreover, the moduli space is defined on a neighborhood of a singular point by at most $\dim H^2(\mathrm{End}_0 E)$ equations. In fact, in the proof of his theorem cited above, Donaldson shows that $\dim H^2(\mathrm{End}_0 E) \leq a+b\sqrt{k}+3k$. Therefore, using Kirwan's result we have that $H_q(\mathfrak{M}_k(S)) = H_q(\mathfrak{M}_k(S) - Sing)$ for $q < \dim\mathfrak{M}_k(S) - 2(a + b\sqrt{k} + 3k) < k$. □

Henceforth we consider only the smooth part of $\mathfrak{M}_k(\widetilde{X})$, which in what follows stands for just the set of its smooth points. Similarly, when considering $\mathfrak{M}_k(X)$ we will consider only its set of smooth points, keeping in mind that this does not change the homology up to dimension k.

3.2 Removing unstable bundles

Given an $\widetilde{\mathcal{L}}$-stable bundle $\widetilde{E}$, the bundle $E = (\pi_*\widetilde{E})^{\vee\vee}$ is $\mathcal{L}$-semistable [FM, Thm. 6.5]. We want to remove the set of strictly semistable bundles that appear in this process, leaving only stable bundles over X, as these are the ones corresponding to instantons.

PROPOSITION 3.2 *Removing semistable bundles that are not stable does not change homology in dimension less than $7k - c$. That is if $\mathfrak{M}^{su}$ denotes the subset of semistable bundles on X that are not stable, then there is a constant c depending only on X such that for $q < 7k - c$*

$$H_q(\mathfrak{M}_k(X)) = H_q(\mathfrak{M}_k(X) - \mathfrak{M}_k^{su}(X)).$$

Proof. Let E be a semistable bundle on X that is not stable. Since $c_1(E) = 0$ there is a destabilising (saturated) line bundle L with $\deg(L) = 0$. Hence, E is an extension

$$0 \to L \to E \to \mathcal{F} \to 0, \tag{3}$$

where $\mathcal{F}$ is a rank 1 torsion free sheaf satisfying $\mathcal{F}^{\vee\vee} = L^{-1}$ and fitting into a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}^{\vee\vee} \to Q \to 0,$$

with Q supported only at points. Taking $\mathrm{Hom}(.,L)$ on the short exact sequence (3) gives

$$\to \mathrm{Ext}^1(Q,L) \to \mathrm{Ext}^1(\mathcal{F}^{\vee\vee},L) \to \mathrm{Ext}^1(\mathcal{F},L) \to \mathrm{Ext}^2(Q,L) \to \mathrm{Ext}^2(\mathcal{F}^{\vee\vee},L) \to .$$

But, because Q is supported at $k = c_2(E)$ points, we have $\mathrm{Hom}(Q,L) = \mathrm{Ext}^1(Q,L) = 0$, and $\mathrm{Ext}^2(Q,L) = \mathbb{C}^k$, and consequently the long exact sequence becomes

$$0 \to H^1(L^2) \to \mathrm{Ext}^1(\mathcal{F}, L) \xrightarrow{\delta} Ext^2(Q, L) = \mathbb{C}^k \to H^2(L^2) \to . \qquad (4)$$

Note that not all sequences of the form (3) result in a bundle E, but as we only consider bundles, we must identify when the middle term is locally free.

Locally we have a resolution

$$0 \to L \to E \to F^{\vee\vee} \to Q \to 0$$

$$0 \to \mathcal{O} \to \mathcal{O}^2 \to \mathcal{O} \to Q \to 0$$

and it follows that Q is a local complete intersection. We write $Q = \mathcal{O}/(p, q)$ with $p, q \in \mathcal{O}$ coprime. Assuming this, we have that $\mathcal{E}xt^1(Q, \mathcal{O}) = 0$ and $\mathcal{E}xt^2(Q, \mathcal{O}) \simeq Q$. The extension (3) has a corresponding class $[E] \in \mathcal{E}xt^1(F, L) \simeq \mathcal{E}xt^2(Q, L) = \mathcal{E}xt^2(Q, \mathcal{O}) \otimes L$. In order for E to be a bundle we need that $[E]$ have a non-zero value at each point in the support of Q. We have

$$\mathrm{Ext}^1(F, L) \to \Gamma\mathcal{E}xt^1(F, L) = \Gamma\mathcal{E}xt^2(Q, L) = \mathrm{Ext}^2(Q, L)$$

and the map from leftmost to rightmost terms in the above expression is just the connecting homomorphism δ coming from

$$0 \to F \to F^{\vee\vee} \to Q \to 0.$$

It follows that the dimension of the set $\mathfrak{M}^{su} := \mathfrak{M}^{ss} - \mathfrak{M}^{s}$ of semistable bundles on X that are not stable is given by

$$\dim\mathfrak{M}^{su}_k = \dim(\delta(\mathrm{Ext}^1(\mathcal{F}, L)) \times \mathrm{Jac}(X))$$

and using the exact sequence (4) we get

$$\dim\mathfrak{M}^{su}_k \leq k + \dim(H^1(L^2) \times \mathrm{Jac}(X)).$$

where $c = \dim(H^1(L^2) \times \mathrm{Jac}(X))$ depends only on X. Now apply Kirwan's theorem with codimension $\dim\mathfrak{M}^{s}_k - \dim\mathfrak{M}^{su}_k = 8k - (1 - b_1 + b^+) - (k + c)$ (singularities were removed in the previous lemma). □

Based on the results of this section, we assume in what follows that the moduli spaces $\mathfrak{M}_i$ are smooth and contain only stable points. We now proceed to the study of framed bundles.

4. Holomorphic instanton patching

In this section we give the detailed construction of holomorphic instanton patching which gives the following result.

PROPOSITION 4.1 *Every (framed) instanton on $\widetilde{X}$ is obtained by holomorphic patching an instanton on X to an instanton on $\widetilde{\mathbb{C}^2}$.*

We give the patching in terms of holomorphic bundles, via the Kobayashi–Hitchin correspondence. For a Kähler surface X, Kobayashi and Hitchin gave a one-to-one correspondence between irreducible $SU(2)$ instantons of charge k on X and stable rank 2 holomorphic bundles E on X with $c_1 = 0$ and $c_2 = k$, see [LT]. For the noncompact surface $X = \widetilde{\mathbb{C}^2}$ this correspondence takes an instanton to a holomorphic bundle with an added trivialisation at infinity [Kn]. We show that a framed bundle on $\widetilde{X}$ is uniquely determined by a pair of framed bundles on X and $\widetilde{\mathbb{C}^2}$.

REMARK: Note that all gluing maps used here for holomorphic patching of framed bundles are homotopic to the identity, this is a consequence of Hartog's theorem as explained in section 2.

We fix a neighborhood $N(\ell)$ of the exceptional divisor inside of $\widetilde{X}$ and write $\widetilde{X} = (\widetilde{X} - \ell) \cup N(\ell)$. Set

$$N^0 := N(\ell) - \ell = (\widetilde{X} - \ell) \cap N(\ell).$$

The blow-up map gives an isomorphism $i_1 \colon \widetilde{X} - \ell \to X - \{x\}$. Based on the results of §1, we we know that moduli of bundles on $N(\ell)$ are isomorphic to moduli of bundle on $\widetilde{\mathbb{C}^2}$. Hence, to simplify the exposition, we can assume that there is an isomorphism $i_2 \colon N(\ell) \to \widetilde{\mathbb{C}^2}$. Over the intersection N^0 we have isomorphisms $N(x) - \{x\} \stackrel{i_1}{\leftarrow} N^0 \stackrel{i_2}{\to} \widetilde{\mathbb{C}^2} - \ell$, which we still denote by the same letters i_1 and i_2. Using this isomorphisms we write framed bundles on $\widetilde{X}$ as pairs of framed bundles on X and $\widetilde{\mathbb{C}^2}$.

LEMMA 4.2 *There is a one-to-one correspondence between algebraic vector bundles on $\widetilde{X} - \ell$ and algebraic vector bundles on X.*

Proof. It is trivial that there is a one-to-one correspondence between bundles on $\widetilde{X} - \ell$ and bundles on $X - \{x\}$, because these are isomorphic surfaces. If E is an algebraic bundle on $X - \{x\}$ it extends uniquely to a bundle on X as follows. One extends E over x as a coherent sheaf $\mathcal{E}$ and then taking double dual one obtains a reflexive sheaf $\mathcal{E}^{\vee\vee}$ over X. Since reflexive sheaves have singularities in codimension 2, it follows that $\mathcal{E}^{\vee\vee}$ is locally free. We have $\mathcal{E}^{\vee\vee}|_{X-\{x\}} \simeq \mathcal{E}|_{X-\{x\}} \simeq E$, so $\mathcal{E}^{\vee\vee}$ gives the vector bundle extension of E. □

DEFINITIONS 4.3

- Let $\pi_F \colon F \to Z$ be a bundle over a surface Z that is trivial over $Z_0 := Z - Y$, where Y is a closed submanifold of Z. Given two pairs $f = (f_1, f_2) \colon Z_0 \to \pi_F^{-1}(Z_0)$ and $g = (g_1, g_2) \colon Z_0 \to \pi_F^{-1}(Z_0)$ of fibrewise linearly independent holomorphic sections of $F|_{Z_0}$, we say that f is *equivalent* to g (written $f \sim g$) if $\phi := g \circ f^{-1} \colon V|_{Z^0} \to V|_{Z^0}$ extends to a holomorphic map $\phi \colon F \to F$ over the entire Z. A *frame* of F over Z_0 is an equivalence class of fibrewise linearly independent holomorphic sections of F over Z_0. The set of such frames

$$Fram(Z_0, F) := Hol(Z_0, SL(2, \mathbb{C}))/\sim$$

carries the quotient topology.

- A *framed* bundle $\widetilde{E}^f$ on $\widetilde{X}$ is a pair consisting of a bundle $\pi_{\widetilde{E}} \colon \widetilde{E} \to \widetilde{X}$ together with a frame of $\widetilde{E}$ over $N^0 := N(\ell) - \ell$.

- A *framed* bundle V^f on $\widetilde{\mathbb{C}^2}$ is a pair consisting of a bundle $\pi_V \colon V \to \widetilde{\mathbb{C}^2}$ together with a frame of V over $\widetilde{\mathbb{C}^2} - \ell$.

- A *framed* bundle E^f on X is a pair consisting of a bundle $E \to X$ together with a frame of E over $N(x) - \{x\}$, where $N(x)$ is a small disc neighborhood of x. We will always consider $N(x) = \pi_{\widetilde{E}}(N(\ell))$.

PROPOSITION 4.4 *An isomorphism class $[\widetilde{E^f}]$ of a framed bundle on $\widetilde{X}$ is uniquely determined by a pair of isomorphism classes of framed bundles $[E^f]$ on X and $[V^f]$ on $\widetilde{\mathbb{C}^2}$. We write $\widetilde{E}^f = (E^f, V^f)$.*

Proof. By construction $\widetilde{E} = E_{\amalg_{(s_1,s_2)=(t_1,t_2)}} V$ is made by identifying the bundles as well as the sections over N^0, so that the bundles satisfy $\widetilde{E}|_{N^0} = i_1^*(E|_{N(x)-\{x\}}) = i_2^*(V|_{\widetilde{\mathbb{C}^2}-\ell})$ and the framing (f_1, f_2) of $\widetilde{E}$ satisfies $(f_1, f_2) = (s_1, s_2) \circ i_1 = (t_1, t_2) \circ i_2$.

Let $\phi\colon E^f \to E'^f$ be an isomorphism such that $\phi\circ(s_1,s_2)=(s_1',s_2')$ and let $\xi\colon V^f \to V'^f$ be an isomorphism such that $\xi\circ(t_1,t_2)=(t_1',t_2')$. We have the following diagram of bundle maps:

$$\begin{array}{ccccccc} \widetilde{E^f}|_{\widetilde{X}-\ell} & \to & E^f & \stackrel{\phi}{\to} & E'^f & \leftarrow & \widetilde{E'^f}|_{\widetilde{X}-\ell} \\ \downarrow \pi_{\widetilde{E}} & & \downarrow \pi_E & & \downarrow \pi_{E'} & & \downarrow \pi_{\widetilde{E'}} \\ \widetilde{X}-\ell & \stackrel{i_1}{\to} & X-\{x\} & = & X-\{x\} & \stackrel{i_1}{\leftarrow} & \widetilde{X}-\ell \end{array}$$

Hence,

$$\widetilde{E'}^f|_{\widetilde{X}-\ell} = i_1^*(E'^f|_{X-\{x\}}) = i_1^*\circ\phi(E^f|_{X-\{x\}}) = i_1^*\circ\phi\circ i_{1*}(\widetilde{E}^f|_{\widetilde{X}-\ell}) \qquad (4)$$

showing that $i_1^*\circ\phi\circ i_{1*}$ is an isomorphism of $\widetilde{E}$ and $\widetilde{E'}$ over $\widetilde{X}-\ell$ such that

$$\phi\circ(f_1,f_2)=\phi\circ(s_1,s_2)\circ i_1=(s_1',s_2')\circ i_1=(f_1',f_2'). \qquad (5)$$

On the other hand we have the second diagram of bundle maps:

$$\begin{array}{ccccccc} \widetilde{E^f}|_{N(\ell)} & \to & V^f & \stackrel{\xi}{\to} & V'^f & \leftarrow & \widetilde{E'^f}|_{N(\ell)} \\ \downarrow \pi_{\widetilde{E}} & & \downarrow \pi_V & & \downarrow \pi_{V'} & & \downarrow \pi_{\widetilde{V'}} \\ N(\ell) & \stackrel{i_2}{\to} & \widetilde{\mathbb{C}^2} & = & \widetilde{\mathbb{C}^2} & \stackrel{i_2}{\leftarrow} & N(\ell) \end{array}.$$

Therefore,

$$\widetilde{E'}^f|_{N(\ell)} = i_2^*(V'^f|_{\widetilde{\mathbb{C}^2}}) = i_2^*\circ\xi(V^f|_{\widetilde{\mathbb{C}^2}}) = i_2^*\circ\xi\circ i_{2*}(\widetilde{E}^f|_{N(\ell)}) \qquad (6),$$

showing that $i_2^*\circ\xi\circ i_{2*}$ is an isomorphism of $\widetilde{E}$ and $\widetilde{E'}$ over $N(\ell)$ such that

$$\xi\circ(f_1,f_2)=\xi\circ(t_1,t_2)\circ i_2=(t_1',t_2')\circ i_2=(f_1',f_2'). \qquad (7)$$

These isomorphisms agree over the intersection N^0, in fact, by (4) and (6)

$$i_1^*\circ\phi\circ i_{1*}(\widetilde{E^f}|_{N^0}) = i_1^*\circ\phi(E^f|_{N(x)}) = i_1^*(E'^f|_{N(x)}) = i_2^*(V'^f|_{\widetilde{\mathbb{C}^2}-\{0\}}) =$$

$$= i_2^*\circ\xi(V^f|_{\widetilde{\mathbb{C}^2}-\{0\}}) = i_2^*\circ\xi\circ i_{2*}(\widetilde{E^f}|_{N^0})$$

and moreover they also preserve the framings over the intersection, since over N^0, we have, by (5) and (7)

$$\phi\circ(f_1,f_2)=(f_1',f_2')=\xi\circ(f_1,f_2).$$

By the gluing lemma this gives an isomorphism over the entire $\widetilde{X}$ and we get $\widetilde{E'} \simeq \widetilde{E}$. □

Note: It is also possible to define framings as being just trivializing sections, without putting the equivalence relation at first, and later divide by automorphisms of the bundle that preserve the framings. However, we found it convenient not to carry on too many inequivalent framings, so that the framed local moduli $\mathcal{N}_i^f$ remain finite dimensional.

5. Framed moduli space of bundles and connections.

Anti-self-dual connections on a connected sum have been extensively studied in the literature, cf. [DK,Chap. 7] and [Ta2] for detailed expositions. In particular, results apply to connections on a blow-up, since differentiably $\widetilde{X} \sim X\#\overline{\mathbb{CP}^2}$. A posteriori, applying the Kobayashi–Hitchin correspondence, one knows that over Kähler surfaces, the theory of stable bundles must be completely parallel to the theory of irreducible ASD connections. Adding frames rigidifies the problem and has the advantage of greatly simplifying gluing constructions. Note that a framing of a bundle induces a framing of the corresponding connection.

To relate anti-self-dual connections on X and $\widetilde{X}$ one needs some compatibility between the metrics $\widetilde{g}$ on $\widetilde{X}$ and g on X. Since anti-self-duality is a conformally invariant condition, it suffices to give a conformal metric $\widetilde{g}$ on $\widetilde{X}$ such that $\widetilde{g}|_{\widetilde{X}-\ell}$ is conformally equivalent to $g|_{X-\{x\}}$. Construction of such conformal metrics is carried out in detail in [TA2, p. 65]. In what follows we consider fixed such metrics. Our proof of the Atiyah–Jones conjecture works for all metrics $\widetilde{g}$ on $\widetilde{X}$ for which the conjecture holds true for the corresponding g on X.

We now recall some well known facts about moduli of vector bundles. Given a C^∞ complex vector bundle F on a compact complex surface Z, a holomorphic structure on F is a $\bar\partial$ operator of type $(0,1)$ which is integrable, hence a holomorphic bundle on Z is a pair $\mathcal{F} = (F, \bar\partial)$; we denote the set of all these by $\mathcal{H}$. Let $\mathcal{G}$ be the group of C^∞-automorphisms of F. Then $\mathcal{G}$ acts on $\mathcal{H}$, and $\bar\partial_1$ and $\bar\partial_2$ are in the same orbit of $\mathcal{G}$ if and only if the corresponding holomorphic bundles $(F, \bar\partial_1)$ and $(F, \bar\partial_2)$ are isomorphic. Given a polarisation H on Z let $\mathcal{H}_k^s(Z) := \mathcal{H}^s(F)$ bet the subset of H-stable bundles (having $c_1 = 0$ and $c_2 = k$); the corresponding moduli space is the quotient $\mathcal{H}_k^s/\mathcal{G}$. By Maruyama's theorem this moduli space is a quasi projective variety [M].

We fix compatible polarisations $\mathcal{L}$ and $\widetilde{\mathcal{L}} = N\mathcal{L} - \ell$ on X and $\widetilde{X}$ through-out as in section 3. Note that this is equivalent to considering the projective embeddings determined by $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ and then, regarding X and $\widetilde{X}$ with the corresponding induced Kähler metrics g and $\widetilde{g}$ to use stability with respect to the polarisations given by the Kähler classes. We denote the moduli spaces of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ stable bundles with $c_1 = 0$ and $c_2 = k$ just by $\mathfrak{M}_k(X)$ and $\mathfrak{M}_k(\widetilde{X})$ respectively. Moreover, using the results of section 3, we removed all singularities, and accordingly we assume that $\mathfrak{M}_k(\widetilde{X})$ and $\mathfrak{M}_k(X)$ are smooth.

Frames are added as in definition 4.3, and an isomorphism between framed bundles (E_1, f^1) and (E_2, f^2) is an isomorphism $\phi \in \mathcal{G}\colon E_1 \to E_2$ taking f^1 to f^2, that is, such that $f^2 = f^1 \circ \phi$. Hence, the complex gauge group acts on framed bundles and the framed moduli spaces are obtained as quotient of such actions. Explicitly, if $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ having $c_1 = 0$ and $c_2 = k$ are the differentiable supports for our holomorphic bundles over X and $\widetilde{X}$, with $\mathcal{G}(\mathcal{E})$ and $\mathcal{G}(\widetilde{\mathcal{E}})$ as their groups of automorphisms, then the framed moduli spaces are:

Definitions 5.1

- $\mathfrak{M}_k^f(\widetilde{X}) := \left\{(\bar{\partial}_{\widetilde{\mathcal{E}}}, f) : f \in Fram\left(N^0, (\widetilde{\mathcal{E}}, \bar{\partial}_{\widetilde{\mathcal{E}}})\right)\right\} / \mathcal{G}(\widetilde{\mathcal{E}})$
- $\mathfrak{M}_k^f(X) := \left\{(\bar{\partial}_{\mathcal{E}}, f) : f \in Fram\left(N(x) - \{x\}, (\mathcal{E}, \bar{\partial}_{\mathcal{E}})\right)\right\} / \mathcal{G}(\mathcal{E})$

Note that $\mathfrak{M}_k^f(\widetilde{X})$ maps to $\mathfrak{M}_k(\widetilde{X})$, and $\mathfrak{M}_k^f(X)$ maps to $\mathfrak{M}_k(X)$ by projection onto the first coordinate. In section 6, we consider the local moduli spaces of bundles defined only over a small tubular neighborhood $N(\ell)$ of the divisor, or what is equivalent, bundles on $\widetilde{\mathbb{C}^2}$. In 6.1 we explicitly calculate dimensions of framed local moduli. This clarifies the important point that the strata appearing in (7) have finite codimension inside the global framed moduli spaces. An independent calculation of codimensions is given in 7.3 as well.

Remark 5.2 We remark that is it possible to choose framings in a different way, so that moduli spaces of framed instantons remain finite dimensional. In fact, by Theorem [G3] bundles on a tubular neighborhood of the exceptional divisor are completely determined by their restriction to a finite infinitesimal neighborhood of order $2j - 2$. Consequently one could choose framings only on such finite infinitesimal neighborhoods and work with finite dimensional framed moduli spaces.

6. Local moduli

For the moduli problem we need to consider the spaces $\mathcal{N}_i^f$ of isomorphism classes of framed bundles on the neighborhood $N(\ell) \simeq \widetilde{\mathbb{C}^2}$ of the exceptional divisor which have fixed charge i (cf. definition 2.1). We define

$$\mathcal{N}_i^f := \{\textit{framed bundles on } N(\ell) \textit{ having charge } i\}/\sim .$$

King [Kn] showed that $\mathcal{N}_i^f$ can be identified with the moduli space $M_i(\Sigma_1)$ of bundles on the first Hirzebruch surface $\Sigma_1 = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O})$ having second Chern class i, and that are trivial (and framed) over the line at infinity ℓ_∞. Using this identification, we show that $M_i(\Sigma_1)$ is smooth and finite dimensional.

This section contains a series of lemmas, which prove the following result.

Theorem 6.1 *$\mathcal{N}_i^f$ is a smooth complex manifold.*

6.1 Dimension of local moduli

Note that bundles on the neighborhood $N(\ell)$ of the exceptional divisor are completely determined by a finite infinitesimal neighborhood of ℓ, consequently the extension class and the inequivalent reframings depend only on a finite number of parameters, hence framed local moduli are finite dimensional.

Proposition 6.2 *Let $F := (j,p)$ be a bundle on $N(\ell)$ with splitting type j and extensions class p, together with a trivialisation on N^0. Suppose $m > 0$ is the u-multiplicity of p (that is, the largest power of u that divides p). Then the dimension of the local moduli space at F is $m(2j - (m+1)/2)$.*

Proof. Following [Kn], we identify $\mathcal{N}_{\mathbf{c}}^f$ with the moduli space $M_{\mathbf{c}}(\Sigma)$ of bundles on the first Hirzebruch surface with $c_1 = 0$ and $c_2 = \mathbf{c}$ with a fixed trivialisation at the line at infinity ℓ_∞. By [L, Thm. 4.6], the Zariski tangent space of $M_{\mathbf{c}}(\Sigma)$ at E is $H^1(\Sigma, \mathrm{End}E \otimes \mathcal{O}_\Sigma(-\ell_\infty))$.

We claim that is $H^1(\Sigma, \mathrm{End}E \otimes \mathcal{O}_\Sigma(-\ell_\infty)) = H^1(N(\ell), EndF)$ where $F = E|_{N(\ell)}$. Clearly $H^1(N(\ell), \mathrm{End}F \otimes \mathcal{O}_\Sigma(-\ell_\infty)) = H^1(N(\ell), \mathrm{End}F)$ because $\mathcal{O}_\Sigma(-\ell_\infty)$ is trivial over $N(\ell)$. We now write $\Sigma = N(\ell) \cup_{N^0} N(\ell_\infty)$ and set $\mathcal{G} = \mathrm{End}E \otimes \mathcal{O}(-\ell_\infty)$. By Mayer-Vietoris

$$H^0(N(\ell), \mathcal{G}) \oplus H^0(N(\ell_\infty), \mathcal{G}) \to H^0(N^0, \mathcal{G}) \to H^1(\Sigma, \mathcal{G}) \to$$

$$\to H^1(N(\ell), \mathcal{G}) \oplus H^1(N(\ell_\infty), \mathcal{G}) \to H^1(N^0, \mathcal{G}).$$

Here $\mathcal{G}|_{N(\ell_\infty)} = \mathcal{O}^{\oplus 4} \otimes \mathcal{O}(-\ell_\infty)$ and $\mathcal{G}|_{N^0} = \mathcal{O}^{\oplus 4}$. It then follows that the map $H^1(N(\ell_\infty), \mathcal{G}) \to H^1(N^0, \mathcal{G})$ is an isomorphism, and also the map $H^0(N(\ell), \mathcal{G}) \to H^0(N^0, \mathcal{G})$ is an isomorphism. The Mayer-Vietoris sequence becomes

$$0 \to H^1(\Sigma, \mathcal{G}) \to H^1(N(\ell), \mathcal{G}) \to 0.$$

It remains to calculate $H^1((N(\ell), \mathcal{G}) = H^1(\mathrm{End}F)$. The transition matrix for $\mathrm{End}F$ in our canonical coordinates is

$$T = \begin{pmatrix} z^{2j} & -pz^j & pz^j & -p^2 \\ 0 & 1 & 0 & pz^{-j} \\ 0 & 0 & 1 & -pz^{-j} \\ 0 & 0 & 0 & z^{-2j} \end{pmatrix}.$$

Because $\mathcal{O}_\Sigma(-\ell_\infty)$ is trivial on $N(\ell)$ the transition matrix for $\mathrm{End}F \otimes \mathcal{O}_\Sigma(-\ell_\infty)$ is the same T. Denote by $\sim$ cohomological equivalence. To compute the H^1 suppose $\sigma = (a, b, c, d)$ is a 1-cocycle. Then it is represented in $U \cap V \simeq \mathbb{C} - \{0\} \times \mathbb{C} = \{(z \neq 0, u)\}$ in the form

$$\sigma = \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} z^l u^i.$$

Since terms having only positive powers of z are holomorphic in U it follows that

$$\sigma \sim \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} z^l u^i.$$

Changing coordinates, that is, calculating $T\sigma$, we get the following conditions for σ to be a coboundary: the expressions

1. $z^{2j}a - pz^j b + pz^j c - p^2 d$
2. $b + pz^{-j} d$
3. $c - pz^{-j} d$
4. $z^{-2j} d$

should be holomorphic in $V = \{z^{-1}, zu)\}$. Expressions 2, 3 and 4 contain no positive powers of z, therefore they are holomorphic in V and impose no extra conditions on σ. The only condition is then that expression 1 be holomorphic in V. Set $p = u^m p'$, then we need to check which terms in the expression

$$z^{2j}a + u^m p' z^j b + u^m p' z^j c + u^{2m} p'^2 d$$

are holomorphic in V, where a, b, c, and d are arbitrary holomorphic in z and u. Choosing b, c, d appropriately we can remove all terms $z^l u^i$ having $i \geq m$. We are let only with

$$z^{2j}a \sim z^{2j} \sum_{i=0}^{m-1} \sum_{l=-\infty}^{-1} a_{il} z^l u^i = \sum_{i=0}^{m-1} \sum_{s=-\infty}^{2j-1} a_{is} z^s u^i.$$

But, since terms having $s \leq i$ are holomorphic in V, the non-zero cocycles come only from the terms $z^s u^i$ with $s > i$, so

$$z^{2j}a \sim \sum_{i=0}^{m-1} \sum_{s=i+1}^{2j-1} a_{is} z^s u^i.$$

Consequently, nontrivial cocycles are represented by sections of the form $\sigma = (a, 0, 0, 0)$ where

$$a \sim \sum_{i=0}^{m-1} \sum_{s=i-2j+1}^{-1} a_{is} z^s u^i.$$

There are $m(2j - (m+1)/2)$ nontrivial coefficients. □

6.2 Smoothness of local moduli.

LEMMA 6.3 *For all E, E' in $\mathcal{N}_i^f$, the map*

$$H^0(\mathrm{Hom}(E, E')) \to H^0(\mathrm{Hom}(E|_{N^0}, E'|_{N^0})$$

is injective.

Proof. N^0 is open in $\widetilde{\mathbb{C}^2}$, hence two holomorphic functions that coincide in N^0 and are globally defined must be equal. □

LEMMA 6.4 *In order to study endomorphisms of a bundle $V \in \mathcal{N}_i$ it suffices to choose a fixed transition matrix T for V and then to consider endomorphisms fixing T.*

Proof. In fact, suppose there is an automorphism ϕ of V taking T to $\bar{T}$. Then, $\phi = (X, Y)$ is given by a pair of transition matrices $X \in \Gamma(U)$ and $Y \in \Gamma(V)$ such that $YTX = \bar{T}$. But, since T and $\bar{T}$ represent the same bundle V, there are change of coordinates $A \in \Gamma(U)$ and $B \in \Gamma(V)$ such that $T = B\bar{T}A$. Now define a new endomorphism of V by $\bar{\phi} = (XA, BY)$, then $\bar{\phi}(T) = BYTXA = B\bar{T}A = T$. □

We use the canonical form of transition matrix $T = \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$ as in (1).

LEMMA 6.5 *Let ϕ be an automorphism of a framed bundle $(V, f) \in \mathcal{N}_i^f$. Then $\phi|_{N^0}$ can be written in the form $\phi(T) = X^{-1}TX$ with $X \in \Gamma(U_0) \cap \Gamma(V_0)$ and $X = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$.*

Proof. That the automorphism can be written as $T \mapsto X^{-1}TX$ follows simply because V is trivial over N^0. Suppose $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. From the equality $XT = TX$ we get $-z^{-j}c = z^j c$, which immediately implies that $c = 0$. □

LEMMA 6.6 *Let ϕ be an automorphism of a framed bundle $(V, f) \in \mathcal{N}_i^f$ over $N(\ell) \simeq \widetilde{\mathbb{C}^2}$. Then ϕ can be written in the form $\phi(T) = X^{-1}TX$ with $X = \begin{pmatrix} a_0 + a & b \\ 0 & d_0 + d \end{pmatrix}$ where $a, b, d \in \Gamma(U_0) \cap \Gamma(V_0)$ with a, b and d vanishing over the exceptional divisor and a_0, d_0 constants.*

Proof. By lemma 6.5 $X = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ over N^0 and we must check what are the possible extensions of X to the full coordinate charts U and V. Let $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ be a holomorphic extension of X to the entire $U-$ chart. We claim the extension is also of the form $X = \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix}$. In fact, $U \simeq \mathbb{C}^2$ and $\bar{c} = c = 0$ on U_0 which is an open subset of U hence $\bar{c} = 0$ everywhere on U. Similarly on V we have the form $X = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ 0 & \bar{\delta} \end{pmatrix}$. Write $\bar{x} = x_0(z) + x(z, u)$ for $x \in \{a, b, d\}$ and write $\bar{y} = \bar{y}_0(z^{-1}) + y(z^{-1}, zu)$ for $y = \{\alpha, \beta, \delta\}$. Over the exceptional divisor one has $p = 0 = u$ and the equality $(*)$ becomes

$$\begin{pmatrix} z^j a_0 & z^j b_0 \\ 0 & z^{-j} d_0 \end{pmatrix} = \begin{pmatrix} z^j \alpha_0 & z^{-j} \beta_0 \\ 0 & z^{-j} \delta_0 \end{pmatrix}$$

where a_0, b_0, d_0 are holomorphic in z whereas $\alpha_0, \beta_0, \delta_0$ are holomorphic in z^{-1}. It immediately follows that $b_0 = \beta_0 = 0$ and that $a_0 = \alpha_0$ and $d_0 = \delta_0$ are constants. □

LEMMA 6.7 *A framed bundle $(V, f) \in \mathcal{N}_i^f$ has no traceless automorphisms, unless V splits.*

Proof. The equality $trace(\lambda X) = \lambda trace(X)$ holds for any $X \in GL(2,\mathbb{C})$ and constant λ. Therefore, the bundle V has traceless $GL(2,\mathbb{C})$ automorphisms if and only if it has traceless $SL(2,\mathbb{C})$ automorphisms. So, we assume $det(X) = 1$. Over the exceptional divisor $X|_\ell = \begin{pmatrix} a_0 & 0 \\ 0 & d_0 \end{pmatrix}$. Hence $a_o d_0 = 1$. If $trace(X)$ is zero, then we also have $a_0 + d_0 = 0$. It follows that $a_0 = \pm i$. Suppose $a_0 = i$, the other case is analogous. Since X is traceless, it then follows that $X = \begin{pmatrix} i+a & b \\ 0 & -i-a \end{pmatrix}$. On the other hand $det(X) = -(i+a)^2 = 1$ implies that either $a = 0$ or $a = -2i$. Therefore $X = \pm\begin{pmatrix} i & b \\ 0 & -i \end{pmatrix}$. Now the equality $XT = TX$ gives

$$\begin{pmatrix} iz^j & ip + z^{-j}b \\ 0 & -iz^{-j} \end{pmatrix} = \begin{pmatrix} iz^j & -ip + z^j b \\ 0 & -iz^{-j} \end{pmatrix}$$

implying

$$2ip = (z^j - z^{-j})b.$$

But b has only positive powers of z, hence the r.h.s. contains powers of z greater of equal to j whereas the l.h.s. only has powers of z strictly smaller than j. Hence $b = 0 = p$. We conclude that the only traceless automorphisms of (V, f) are multiples of $\begin{pmatrix} i & 0 \\ 0 & -i. \end{pmatrix}$. □

PROPOSITION 6.8 *$M_i(\Sigma_1)$ is smooth.*

Proof. By [L] Theorem 1.1, given the conditions of lemma 6.2 above, $M_i(\Sigma_1)$ is smooth at E provided that $H^2(sl(E) \otimes \mathcal{O}_{\Sigma_1}(-\ell_\infty)) = 0$, where $sl(E)$ is the bundle of traceless endomorphisms of E. But lemma 6.7 implies that there are no traceless endomorphisms of E, unless E splits, in which case $sl(E)$ is trivial. The same conclusions then holds also for bundles on $M(\Sigma_1)$. By Serre duality,

$$H^2(sl(E) \otimes \mathcal{O}_{\Sigma_1}(-\ell_\infty)) = H^0(sl(E) \otimes \mathcal{O}_{\Sigma_1}(\ell_\infty) \otimes K).$$

Since $K \simeq \mathcal{O}(-2f - \ell - \ell_\infty)$ where f (having $f^2 = 0$) is the class of the fiber of Σ_1 and $\ell^2 = -1$. Then $\mathcal{O}_{\Sigma_1}(\ell_\infty) \otimes K \simeq \mathcal{O}(-2f - \ell)$, and consequently

$$H^0(sl(E) \otimes \mathcal{O}_{\Sigma_1}(\ell_\infty) \otimes K) = H^0(sl(E)(-2f - \ell))$$

which we claim vanishes. In fact, suppose not, then a global section gives is an injection

$$0 \to \mathcal{O} \to sl(E) \otimes \mathcal{O}(-2f - \ell)$$

and a corresponding short exact sequence

$$0 \to \mathcal{O}(2f+\ell) \to sl(E) \to Q \to 0.$$

Taking cohomology gives

$$0 \to H^0(\mathcal{O}(2f+\ell)) \to H^0(sl(E)) \to H^0(Q) \to$$

But $H^0(\mathcal{O}(2f+\ell)) \neq 0$ whereas by lemma 6.7 $H^0(sl(E)) = 0$, a contradiction. □

7. Proof of the conjecture

We assume that the Atiyah–Jones conjecture holds true for X, that is, we assume that there exist maps $r_k \colon \mathfrak{M}_k^f(X) \to \mathfrak{M}_{k+1}^f(X)$ inducing isomorphisms in homology $r_{k*} \colon H_q(\mathfrak{M}_k^f(X)) \to H_q(\mathfrak{M}_{k+1}^f(X))$ for $q \leq \lfloor k/2 \rfloor - c$ where c is a constant depending on X.

7.1 Statement of the conjecture

For a 4-manifold X, let $\mathcal{M}I_k(X)$ denote the moduli space of framed $SU(2)$ instantons on E with charge k and let $\mathcal{B}_k^\epsilon(X)$ denote the space of framed gauge equivalence classes of connections on X with charge k whose self-dual part of the curvature has norm less than ϵ. Given a point $x_0 \in X$, by patching a small instanton on a neighborhood of x_0, Taubes [Ta] constructed smooth maps

$$t_k^{x_0} \colon \mathcal{M}I_k(X) \to \mathcal{B}_{k+1}^\epsilon(X).$$

He also constructed strong deformation retracts

$$\tau_{k+1} \colon \mathcal{B}_{k+1}^\epsilon(X) \to \mathcal{M}I_{k+1}(X).$$

He then showed that the stable limit $\lim\limits_{k\to\infty} \mathcal{M}I_k$ indeed has the homotopy type of $\mathcal{B}(X)$, the space of framed gauge equivalence classes of connections on E. Consequently, the Atiyah–Jones conjecture in homology is equivalent to the statement that the maps $i_k = \tau_{k+1} \circ t_k^{x_0}$ induce isomorphisms in homology

$$i_{k*} \colon H_q(\mathcal{M}I_k(X)) \to H_q(\mathcal{M}I_{k+1}(X))$$

through a range $q(k)$ increasing with k.

For a compact Kähler surface Z, we denote by

$$\mathcal{K}_X \colon \mathcal{M}I_k(Z) \to \mathfrak{M}_k(Z)$$

the Kobayashi–Hitchin map given by $\mathcal{K}(\nabla = \partial + \overline{\partial}) = \overline{\partial}$, and let $\mathcal{H}_Z := \mathcal{K}_Z^{-1}$ be the inverse map. These maps induce real analytic isomorphisms of moduli spaces, cf. [LT]. Using this translation to moduli of bundles, we aim to prove that $H_q(\mathfrak{M}_k^f(\widetilde{X})) = H_q(\mathfrak{M}_{k+1}^f(\widetilde{X}))$ for $q \leq \lfloor k/2 \rfloor - c$, assuming the analogous statement is true for X. We now stratify $\mathfrak{M}_k^f(\widetilde{X})$ and show that the composite

$$\mathbf{t}_k \colon \mathfrak{M}_k^f(\widetilde{X}) \to \mathfrak{M}_{k+1}^f(\widetilde{X})$$

given by

$$\mathbf{t}_k \colon = \mathcal{K}_{\widetilde{X}} \circ \tau_{k+1} \circ t_k^{\widetilde{x}_0} \circ \mathcal{H}_{\widetilde{X}}$$

is homotopy equivalent to a map that preserves the stratifications.

7.2 Stratifications

We consider stratifications of the moduli spaces which induce filtrations with an associated Leray spectral sequence. We refer to those as L-stratifications.

DEFINITION A smooth manifold M is L-stratified if there is a decomposition of M into disjoint submanifolds $M(K)$ such that
(1) The index set $\mathcal{K} = \{K\}$ is finite with a given fixed well ordering $\leq$.
(2) If K_0 is the smallest element in $(\mathcal{K}, \leq)$, then $M(K_0)$ is an open-dense subset of M.
(3) For all $K \in \mathcal{K}$ the union of the submanifolds of the same or smaller order

$$Z(K) = \cup_{K' \leq K} M(K')$$

is an open-dense submanifold of M.
(4) For all $K \in \mathcal{K}$ the normal bundle, $\nu(K)$, of $M(K)$ in M is orientable.

PROPOSITION 7.2 *There is an L-stratification of the moduli space of framed bundles on $\widetilde{X}$ as*

$$\mathfrak{M}_k^f(\widetilde{X}) \simeq \bigcup_{i=0}^{k} \mathfrak{M}_{k-i}^f(X) \times \mathcal{N}_i^f. \tag{8}$$

Proof. The existence of the point set decomposition follows directly from the definition of the $\mathcal{N}_i$ together with proposition 4.4. The space $K_i = \mathfrak{M}_{k-i}^f(X) \times \mathcal{N}_i^f$, having constant Euler characteristic in each factor, is flat as a family of bundles. Because the moduli of framed stable bundles $\mathfrak{M}_k^f(\widetilde{X})$

is fine (cf. [HL, 4.B]) the flat family $\mathfrak{M}^f_{k-i}(X) \times \mathcal{N}^f_i$ must be obtained by pulling back the universal bundle on $\mathfrak{M}^f_k(\widetilde{X})$ after possibly twisting by a line bundle. But, comparing with the maps appearing in proposition 4.4, shows that the twisting can chosen to be trivial. Lemma 7.3 below shows that the inclusion $K_i \hookrightarrow \mathfrak{M}^f_k(\widetilde{X})$ is an immersion.

The first stratum $K_0 = \mathfrak{M}^f_k(X) \times \mathcal{N}^f_0$ equals the set of pull-back bundles and is dense in $\mathfrak{M}^f_k(\widetilde{X})$. To show this gives an L-stratification we need to prove that the normal bundle of each stratum $K_i = \mathfrak{M}^f_{k-i}(X) \times \mathcal{N}^f_i$ is orientable. Using the results from sections 3.1 and 3.2, we may assume that $\mathfrak{M}^f_k(\widetilde{X})$ is smooth and that $\mathfrak{M}^f_i(X)$ is smooth and contains only stable bundles. By theorem 6.1 $\mathcal{N}^f_i$ is a complex manifold. It then follows that each stratum K_i is a complex submanifold of $\mathfrak{M}^f_k(\widetilde{X})$ and therefore has orientable normal bundle. The remaining properties of the $L-$stratification are shown in the following lemma. □

Lemma 7.3 The inclusion $K_i \hookrightarrow \mathfrak{M}^f_k(\widetilde{X})$ is an immersion.

Proof. Let $\widetilde{E} = (E, V, \phi)$ be image of the pair $((E, h), (V, g))$ where $\phi = h \circ g^{-1}$ is the composition of the framings $h\colon N^0 \to E$ and $g\colon N^0 \to V$. We want to show that the map on tangent spaces

$$T_E\mathfrak{M}^f_{k-i}(X) \times T_V\mathcal{N}^f_i \to T_{\widetilde{E}}\mathfrak{M}^f_k(\widetilde{X})$$

is injective. Now $X = X^0 \amalg_{N(n)-\{x\}} N(x)$ and we have the exact sequences

$$0 \to \frac{H^0(N^0, \mathrm{End}E)}{H^0(N(x), \mathrm{End}E)} \to T_{\mathfrak{M}^f_{k-i}(X),(E,h)} \to H^1(X, \mathrm{End}E)$$

and

$$0 \to \frac{H^0(N^0, \mathrm{End}V)}{H^0(N(\ell), \mathrm{End}V)} \to T_{N(\ell),(V,g)} \to H^1(N(\ell), \mathrm{End}V).$$

Note that, by construction, the moduli of pairs $((E, h), (V, g))$ is the same as the moduli of triples (E, V, ϕ) plus the moduli of framings, g. But, by Hartog's we have that $\frac{H^0(N^0,\mathrm{End}E)}{H^0(N(x),\mathrm{End}E)} = 0$, from which it follows that the map on tangent spaces is injective. □

For each $0 \le n \le k$, set $S_n = \cup_{i \ge n} K_i$ and $Z_n = \mathfrak{M}^f_k(\widetilde{X}) \setminus S_n$.

Lemma 7.4 *For all n, Z_n is open and dense in $\mathfrak{M}^f_k(\widetilde{X})$. The real codimension of the stratum K_i in $\mathfrak{M}^f_k(\widetilde{X})$ is at least $2i$.*

Proof. By definition $K_i = \mathfrak{M}^f_{k-i}(X) \times \mathcal{N}^f_i$. The first stratum is the set of pull-back bundles $K_0 = \{\pi^*(E), E \in \mathfrak{M}^f_k(X)\}$ is non-empty we will se below that it is open and dense in $\mathfrak{M}^f_k(\widetilde{X})$. It follows that $S_1 = K_1 \cup K_2 \cdots \cup K_k$ has (complex) codimension 1, in particular K_1 has codimension at least 1.

For $F \in \mathfrak{M}_k(\widetilde{X})$, set $\mathbf{r}(F) := (R^1\pi_*F)_x$ and $\mathbf{q}(F) := ((\pi_*F)^{\vee\vee}/\pi_*F)_x$ denote the stalks of the first derived image and of the quotient sheaf Q at x; and consider the sets

$$\mathbf{R}_n := \{F \in \mathfrak{M}^f_k(X)\}, h^0(\mathbf{r}(F)) \geq n\}$$

and

$$\mathbf{Q}_n := \{F \in \mathfrak{M}^f_k(X)\}, h^0(\mathbf{q}(F)) \geq n\}.$$

We consider the morphism $f\colon \mathfrak{M}^f_k(X) \to \{x\}$. Since the target is just a point, any sheaf on $\mathfrak{M}^f_k(X)$ is f-flat. Since bundles in $\mathfrak{M}^f_k(X)$ are framed stable, by [HL, thm. 4.B.4] we know that there exists a universal sheaf $\mathcal{U}$ over $\mathfrak{M}^f_k(X)$. Regarding $\mathbf{r}$ and $\mathbf{h}$ as functions applied to $\mathcal{U}$, it then follows from Grauert's semicontinuity theorem [Ha, p. 288] that the functions $h^0(x, \mathbf{r}(\mathcal{U}))$ and $h^0(x, \mathbf{h}(\mathcal{U}))$ are upper-semicontinuous. Hence for each n, the sets $\mathbf{R}_n$ and $\mathbf{Q}_n$ are closed in $\mathfrak{M}^f_k(X)$.

Now set $r_0 := \mathbf{r}$, $q_0 := \mathbf{q}$, and for $1 \leq n \leq k$, set $r_n := \mathbf{r}|_{\mathbf{R}_{n-1}}$ and $q_n := \mathbf{q}|_{\mathbf{Q}_{n-1}}$. Repeating the above reasoning for r_1 and q_1, we get that $S_2 = \mathbf{R}_1 \cup \mathbf{Q}_1$ (by thm 0.2 in [BG1]) is closed in S_1. It follows that the codimension of K_2 in $\mathfrak{M}^f_k(X)$ is at least 2. Now use induction on n. □

7.3 Maps between the spectral sequences

Here we show that the map $\mathbf{t}_k\colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_{k+1}(\widetilde{X})$ is homotopy equivalent to a map that preserves the stratifications $\mathfrak{M}^f_k(\widetilde{X}) = \cup K_i$ and $\mathfrak{M}^f_{k+1}(\widetilde{X}) = \cup K'_i$ where $K_i = \mathfrak{M}^f_{k-i}(X) \times \mathcal{N}^f_i$ and $K'_i = \mathfrak{M}^f_{k-i+1}(X) \times \mathcal{N}^f_i$ as in (8). Let

$$d\colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_k(X \amalg_f \widetilde{\mathbb{C}^2})$$

be the map that re-writes a framed bundle $\widetilde{E}^f$ on $\widetilde{X}$ into its two components of the decomposition $d(\widetilde{E}^f) = (E^f, V^f)$ where E^f is a framed bundle on X and V^f is a framed bundle on $\widetilde{\mathbb{C}^2}$.

Remark 7.4: By proposition 4.4, d is a bijection, and the set of pairs $\{(E^f, V^f)\}$ is given the topology induced by this bijection. In fact, d is just fancy way of expressing the identity map in a form that is convenient for our constructions; in particular, d is a homeomorphism.

We choose a point x_0 in X that is far from the point x we blew-up, in the sense that $x_0 \notin \pi(N(\ell))$ and we set $\widetilde{x_0} = \pi^{-1}(x_0)$. Let

$$t_k^{x_0} \colon \mathcal{M}I_k(X) \to \mathcal{B}^{\epsilon}_{k+1}(X) \text{ and } t_k^{\widetilde{x_0}} \colon \mathcal{M}I_k(\widetilde{X}) \to \mathcal{B}^{\epsilon}_{k+1}(\widetilde{X})$$

denote Taubes patching on a small neighborhood of x_0 in X and of $\widetilde{x}_0$ in $\widetilde{X}$ respectively, and let

$$\tau_{k+1} \colon \mathcal{B}^{\epsilon}_{k+1}(X) \to \mathcal{M}I_{k+1}(X) \text{ and } \widetilde{\tau}_{k+1} \colon \mathcal{B}^{\epsilon}_{k+1}(\widetilde{X}) \to \mathcal{M}I_{k+1}(\widetilde{X})$$

be the corresponding deformation retracts. The idea is that by choosing x to be far from the exceptional divisor, we keep the patching far from the exceptional divisor as well. From the point of view of vector bundles, this implies that we can choose to increase the second Chern class in such a way that it does not alter the bundle near the divisor. We now make this statement precise.

Proposition 7.5 *The maps* $\mathbf{t}_k \colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_{k+1}(\widetilde{X})$ *and* $\mathbf{r}_k \colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_{k+1}(\widetilde{X})$ *given by*

$$\mathbf{t}_k \colon = \mathcal{K}_{\widetilde{X}} \circ \tau_{k+1} \circ t_k^{\widetilde{x}_0} \circ \mathcal{H}_{\widetilde{X}}$$

and

$$\mathbf{r}_k \colon = d^{-1} \circ (\mathcal{K}_X, \mathcal{K}_{\widetilde{\mathbb{C}}^2}) \circ \tau_{k+1} \circ (t_k^{x_0}, id) \circ (\mathcal{H}_X, \mathcal{H}_{\widetilde{\mathbb{C}^2}}) \circ d$$

are homotopically equivalent. Moreover, the map $\mathbf{r}_k$ *preserves the stratification.*

Proof. The following diagram summarises the situation

$$\begin{array}{ccccccccccc}
\mathfrak{M}^f_k(\widetilde{X}) & \overset{\mathcal{H}_{\widetilde{X}}}{\longrightarrow} & \mathcal{M}I^f_k(\widetilde{X}) & \overset{t_k^{\tilde{x}_0}}{\longrightarrow} & \mathcal{B}^{\epsilon}_{k+1}(\widetilde{X}) & \overset{\tilde{\tau}_{k+1}}{\longrightarrow} & \mathcal{M}I^f_{k+1}(\widetilde{X}) & \overset{\mathcal{K}_{\widetilde{X}}}{\longrightarrow} & \mathfrak{M}^f_{k+1}(\widetilde{X}) \\
d \downarrow & & \wr| & & \wr| & & \wr| & & d^{-1} \uparrow \\
\mathfrak{M}^f_k(\bar{X}) & \overset{(\mathcal{H}_X, \mathcal{H}_{\widetilde{\mathbb{C}}^2})}{\longrightarrow} & \mathcal{M}I^f_k(\bar{X}) & \overset{(t_k^{x_0}, id)}{\longrightarrow} & \mathcal{B}^{\epsilon}_{k+1}(\bar{X}) & \overset{\tau_{k+1}}{\longrightarrow} & \mathcal{M}I^f_{k+1}(\bar{X}) & \overset{(\mathcal{K}_X, \mathcal{K}_{\widetilde{\mathbb{C}}^2})}{\longrightarrow} & \mathfrak{M}^f_{k+1}(\bar{X})
\end{array} .$$

In the second row $\bar{X}$ stands for $\bar{X} := (X - \{x\})_{\amalg_{i_1 = i_2}} \widetilde{\mathbb{C}^2}$ identified as in section 4. A bundle on $\bar{X}$ given by a pair (E, V) of bundles on X and $\widetilde{\mathbb{C}^2}$ again as in section 4. Note that each of the vertical maps is a bijection. It is important to keep in mind that all framings are given on open sets, not just at points. The maps $\mathcal{H}, \mathcal{K}, \tau$ and t are well known to be continuous, cf.

[LT] and [Ta2]. Note that the homeomorphism $d(\widetilde{E}^f) = (E^f, V^f)$ satisfies $E = \pi_* \widetilde{E}^{\vee\vee}$ and $V \simeq \widetilde{E}|_{N(\ell)}$; in particular, $E|_{N^0} \simeq V|_{N^0}$ are identified via the framings (see Remark 7.4), that is, there are isomorphisms of framed bundles

$$E^f|_{N^0} \xleftarrow{i_1} \widetilde{E}^f|_{N^0} \xrightarrow{i_2} V^f|_{N^0}. \tag{8}$$

Moreover, $\mathcal{H}_X|_{N^0}$ takes the $\bar{\partial}_E$ operator over N^0 to the unique $SU(2)$ connection ∇_E on $E|_{N^0}$ having this $\bar{\partial}_E$ as its $(0,1)$ part. Uniqueness is proven as in [DK, Lemma 2.1.54]. For our purposes it is important to notice that the proof of this lemma is carried out on local trivialisations, uniqueness being obtained in each coordinate patch. Explicitly, the matrix of 1-forms α_E^τ giving the partial connection $\bar{\partial}_E$ is associated to the connection $\nabla_E^\tau = \alpha_E^\tau - (\alpha^\tau)_E^*$. Similarly, $\mathcal{H}_{\widetilde{\mathbb{C}}^2}|_{N^0}$ takes the $\bar{\partial}_V$ operator over N^0 to the unique $SU(2)$ Hermitian Yang–Mills connection ∇_V on $V|_{N^0}$ having this $\bar{\partial}_V$ as its $(0,1)$ part. The framings on bundles induce framings on the corresponding connections, and the isomorphisms of framed bundles i_1 and i_2 in (8) induce isomorphisms of framed connections

$$\nabla_E^f|_{N^0} \xleftarrow{i_1} \nabla_{\widetilde{E}}^f|_{N^0} \xrightarrow{i_2} \nabla_V^f|_{N^0}. \tag{9}$$

It follows that $(\mathcal{H}_X, \mathcal{H}_{\widetilde{\mathbb{C}}^2})$ is well defined on the moduli space (that is, on isomorphism classes) and continuous; and we have commutativity of the first square. A similar reasoning shows commutativity of the last square.

In the second square, we chose $t_k^{x_0}$ to be Taubes patching of a nearly anti-self-dual connection around a point $x_0 \notin N(\ell)$. Since the patching is a local operation, we can chose it so that it does not change the connection on $N(\ell)$, that is, we may assume that $(t_k^{x_0}\nabla)|_{N(\ell)} = \nabla|_{N(\ell)}$ and we can frame $(t_k^{x_0}\nabla)$ accordingly over N^0. Hence, there is an identification $(t_k^{x_0}\nabla) \simeq \nabla$ over N^0 via the framings and it follows that $(t_k^{x_0}, id)$ is well defined and continuous. Using the isomorphisms given by the vertical maps, we may consider the maps $t := (t_k^{x_0}, id)$ and $\tilde{t} := t_k^{\tilde{x}_0}$ as being both defined on the same spaces $\mathcal{MI}_k^f(\widetilde{X}) \to \mathcal{B}_{k+1}^\epsilon(\widetilde{X})$. For any instanton $\nabla \in \mathcal{MI}_k(\widetilde{X})$, the self-dual parts of the curvatures of $\tilde{t}(\nabla)$ and $t(\nabla)$ have L^2 norm less than ϵ. Note also that t and $\tilde{t}$ can be chosen to change the connection only on a small ball $U(x_0)$ around x_0. We then use the norm of the curvature to estimate the norm of the connections over $U(x_0)$, as in [DK, §2.3]. This shows that $t(\nabla)$ and $\tilde{t}(\nabla)$ are at distance less than 2ϵ on $U(x_0)$ and by construction, they coincide outside $U(x_0)$. It follows that the L^2 distance between t and $\tilde{t}$ is less than 2ϵ. Choosing ϵ small enough, this implies that t and $\tilde{t}$ are homotopic.

In the third square of our diagram, the maps τ_{k+1} and $\tilde{\tau}_{k+1}$ are deformation retractions. Note that the map τ_{k+1} is applied to connections of the form $\nabla \in im(t_k^{x_0}, id)$, where ∇ is anti-self-dual over $N(\ell) \simeq \widetilde{\mathbb{C}^2}$, and framed over N^0. The point here is to guaranty that the image consists of connections that are also framed on N^0, so the image falls in $\mathcal{M}I^f_{k+1}(\bar{X})$. In fact, this follows from the definition of τ_{k+1} because this deformation retraction is constructed via an application of the contraction mapping theorem; consequently, $im\tau_{k+1}$ consists of instantons that are also framed on N^0. There is no reason to expect that the vertical isomorphisms make the diagram commutative, because there is no obvious way to compare $im\tilde{\tau}_{k+1}$ to $im\tau_{k+1}$. However, the horizontal maps are deformation retractions, and the vertical maps are isomorphisms, so the third square homotopy commutes; and this is all we need. □

7.6 Proof of AJ for blow-ups

Let $\mathfrak{S}_k$ denote the Leray spectral sequence associated to the stratification $\{K_i\}$ of $\mathfrak{M}^f_k(\widetilde{X})$ and let $\mathfrak{S}'_{k+1}$ denote the Leray spectral sequence associated to the stratification $\{K'_i\}$ of $\mathfrak{M}^f_{k+1}(\widetilde{X})$.

Theorem 7.6 *The maps* $\mathbf{t}_k \colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_{k+1}(\widetilde{X})$ *induce isomorphisms in homology* $H_q(\mathfrak{M}^f_k(\widetilde{X})) \to H_q(\mathfrak{M}^f_{k+1}(\widetilde{X}))$ *for* $q \leq \lfloor k/2 \rfloor - c$.

Proof. By proposition 7.5 we may assume that $\mathbf{t}_k \colon \mathfrak{M}^f_k(\widetilde{X}) \to \mathfrak{M}^f_{k+1}(\widetilde{X})$ respects the stratifications and therefore induces a map of spectral sequences $\mathbf{t}_k \colon \mathfrak{S}_k \to \mathfrak{S}'_{k+1}$. The E^1 term of $\mathfrak{S}_k$ is $E^1_{p,q} = H_p(T\nu(K_q))$, where $T\nu$ denotes the Thom space of the normal bundle, and the E^1 term of $\mathfrak{S}'_{k+1}$ is $E^1_{p,q} = H_p(T\nu(K'_q))$. By hypothesis Atiyah–Jones holds for X, hence we are assuming that for each k and for $q \leq \lfloor k/2 \rfloor - c$, $H_q(\mathfrak{M}^f_k(X)) = H_q(\mathfrak{M}^f_{k+1}(X))$. Consequently

$$H_q(K_i) = H_q(\mathfrak{M}^f_{k-i}(X) \times \mathcal{N}^f_i) = H_q(\mathfrak{M}^f_{k-i+1}(X) \times \mathcal{N}^f_i) = H_q(K'_i)$$

for $q \leq \lfloor (k-i)/2 \rfloor - c$. The Thom isomorphism gives $H_*(T\nu(K_i)) = H_{*-\tau(K_i)}(K_i)$, where $\tau(K_i)$ is the real codimension of K_i and similarly for K'_i. By lemma 7.4 the real codimension of K_i at least $2i$. Therefore

$$H_q(T\nu(K_i)) = H_q(T\nu(K'_i)) \ \text{for}\, q \leq \lfloor (k-i)/2 \rfloor - c + 2i.$$

The E^1 terms are:

$\cdots$	$\cdots$	$\cdots$	
$H_2(T_\nu K_0)$	$H_2(T_\nu(K_1)$	$H_2(T_\nu(K_2)$	$\cdots$
$H_1(T_\nu K_0)$	$H_1(T_\nu(K_1)$	$H_1(T_\nu(K_2)$	$\cdots$
$H_0(T_\nu K_0)$	$H_0(T_\nu(K_1)$	$H_0(T_\nu(K_2)$	$\cdots$

Since $\lfloor (k-i)/2 \rfloor - c + 2i \geq \lfloor k/2 \rfloor - c$, we conclude that $\mathbf{t}_k \colon \mathfrak{S}_k \to \mathfrak{S}'_{k+1}$ induces an isomorphism of E^1 terms for all $E^1_{r,s}$ with $r+s \leq \lfloor k/2 \rfloor - c$. □

Corollary 7.7 *The Atiyah–Jones conjecture is true for rational surfaces.*

Proof. Every rational surface is obtained by blowing up points on $\mathbb{P}^2$ or on a rational ruled surface, and on these cases the conjecture holds true by [BHMM] and [HM]. □

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