CHERN CLASSES OF BUNDLES ON BLOWN–UP SURFACES

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Abstract. Consider the blow up $\pi\colon \widetilde{X} \to X$ of a complex surface X at a point. Let $\widetilde{E}$ be a holomorphic bundle over $\widetilde{X}$ whose restriction to the exceptional divisor is $\mathcal{O}(j) \oplus \mathcal{O}(-j)$ and define $E = (\pi_* \widetilde{E})^{\vee\vee}$. Friedman and Morgan gave the following bounds for the second Chern classes $j \le c_2(\widetilde{E}) - c_2(E) \le j^2$. We show that these bounds are sharp.

1. Introduction

We consider a complex surface X together with the blow–up $\pi\colon \widetilde{X} \to X$ of a point $x \in X$ and we denote by ℓ the exceptional divisor. Let $\widetilde{E}$ be a rank–2 holomorphic bundle over $\widetilde{X}$ satisfying $\det \widetilde{E} \simeq \mathcal{O}_{\widetilde{X}}$. The splitting type of $\widetilde{E}$ is by definition the integer $j \ge 0$ such that $\widetilde{E}|_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j)$. Set $E = \pi_* \widetilde{E}^{\vee\vee}$. Assuming X compact, Friedman and Morgan [3, p. 393] gave the following estimate relating the second Chern classes to the splitting type

$$j \le c_2(\widetilde{E}) - c_2(E) \le j^2. \tag{1.1}$$

We show that these bounds are sharp. We calculate the difference $c_2(\widetilde{E}) - c_2(E)$ starting from the transition matrix defining $\widetilde{E}$ on a neighborhood of ℓ. Our methods are very concrete and our calculations make essential use of the Theorem on Formal Functions [4, p. 277].

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The author's motivation to check sharpness of these bounds came from the study of instantons in mathematical physics, where the number $c_2(\widetilde{E}) - c_2(E)$ can be looked upon as the local contribution to the charge obtained by patching a $\widetilde{\mathbb{C}^2}$–instanton to an instanton on X (for a mathematical treatment of instanton patching see [10, Thm.2]). The translation from mathematical physics to algebraic geometry is made through the well–known Hitchin–Kobayiashi correspondence (see, e.g., [9] and references therein); which associates instantons to holomorphic bundles. In particular, to an instanton on $\widetilde{\mathbb{C}^2}$ it associates a holomorphic bundle together with a trivialization at infinity. We describe bundles on $\widetilde{\mathbb{C}^2}$ by explicitly giving their transition matrices, therefore their existence is guaranteed and hence the existence of the corresponding instanton. For each compact complex surface $\widetilde{X}$ we give bundles on the blow-up $\widetilde{X}$ which, on a neighborhood of the exceptional divisor, are also explicitly described by transition matrices; and outside of the exceptional divisor are a pull-back, therefore their existence is also guaranteed. Applications to the construction of instantons will appear elsewhere. Here we will use the term *charge* of $\widetilde{E}$ for $c_2(\widetilde{E}) - c_2(E)$, but will restrict ourselves to the algebraic–geometric calculations.

Let F be a bundle on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class. If X is a compact complex surface, then there exist holomorphic bundles $\widetilde{E} \to \widetilde{X}$ which are isomorphic to F on a neighborhood of the exceptional divisor. In fact, following [8], given a bundle $E \to X$, we can construct bundles $\widetilde{E} \to \widetilde{X}$ satisfying:

ι) $\widetilde{E}|_{\widetilde{X}-\ell} = \pi^*(E|_{X-p})$, where π is the blow–up map, and

$\iota\iota$) $\widetilde{E}|_V \simeq F|_U$ for small neighborhoods V and U of the exceptional divisor in $\widetilde{X}$ and $\widetilde{\mathbb{C}^2}$ respectively.

Moreover, every bundle $\widetilde{E}$ is obtained this way (see [8, Cor. 3.4]). The isomorphism class of $\widetilde{E}$ depends on the attaching map $\phi \colon (\widetilde{X} - \ell) \cap V \to GL(2, \mathbb{C})$, however, the topological type of $\widetilde{E}$ is independent of ϕ (see [8, Cor. 4.1]). Therefore the charge does not depend upon the choice of ϕ. Since the $n-$th infinitesimal neighborhood of the exceptional divisor on a compact complex surface is isomorphic (as a scheme) to the $n-$th infinitesimal neighborhood of the exceptional divisor on $\widetilde{\mathbb{C}^2}$ we are able to use an explicit description for bundles on $\widetilde{\mathbb{C}^2}$ to calculate the charge of $\widetilde{E}$.

For bundles on the blow–up of $\mathbb{C}^2$, we fix, once and for all, the following charts: $\widetilde{\mathbb{C}^2} = U \cup V$ where $U = \{(z, u)\} \simeq \mathbb{C}^2 \simeq \{(\xi, v)\} = V$ with $(\xi, v) = (z^{-1}, zu)$ in $U \cap V$. Once these charts are fixed, there is a canonical choice of transition matrix for bundles on $\widetilde{\mathbb{C}^2}$; namely, we recall from [7, Thm. 2.1], that *if E is a holomorphic bundle on $\widetilde{\mathbb{C}^2}$ with $E|_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j)$, then*

E has a transition matrix of the form

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \tag{1.2}$$

from U to V, where

$$p := \sum_{i=1}^{2j-2} \sum_{l=i-j+1}^{j-1} p_{il} z^l u^i \tag{1.3}$$

is a polynomial in z, z^{-1} and u.

In other words, a holomorphic rank–2 bundle $\widetilde{E}$ over $\widetilde{\mathbb{C}^2}$ with trivial determinant is *algebraic*, and can be written as an extension of the form,

$$0 \to \mathcal{O}(-j) \to \widetilde{E} \to \mathcal{O}(j) \to 0 \tag{1.4}$$

with extension class $p \in Ext^1(\mathcal{O}(j), \mathcal{O}(-j))$ given by a polynomial p of the form (1.3). Moreover, since the equation of the exceptional divisor ℓ on the U–chart is $u = 0$, we see from the expression of p that $\widetilde{E}$ is completely determined by its restriction to the $(2j-2)$nd formal neighborhood of ℓ which we denote by N_ℓ.

If we now consider the blow–up $\widetilde{X}$ at a point on a *compact* complex surface X, then *every holomorphic rank–2 vector $\widetilde{E}$ bundle over $\widetilde{X}$ with vanishing first Chern class is topologically determined by a triple (E, j, p) where E is a rank–2 holomorphic bundle on X with vanishing first Chern class, j is a non–negative integer, and p is a polynomial* (see [8, Cor. 4.1]); in this case, we write

$$\widetilde{E} := (E, j, p). \tag{1.5}$$

The pair (j, p) gives an explicit description of $\widetilde{E}$ on a neighborhood of the exceptional divisor, and determines the charge of $\widetilde{E}$. We are going to calculate numerical invariants for the cases $p = u$ and $p = 0$, thus showing that the bounds on (1.1) are sharp.

We actually calculate two finer numerical invariants of $\widetilde{E}$, which we now describe. Following Friedman and Morgan [3, p. 302], we define a sheaf Q by the exact sequence,

$$0 \to \pi_* \widetilde{E} \to E \to Q \to 0. \tag{1.6}$$

Note that Q is supported only at the point x. From the exact sequence (1.6) it follows immediately that $c_2\pi_*(\widetilde{E}) - c_2(E) = l(Q)$, where l stands for length. An application of Grothendieck–Riemann–Roch (see [3, p. 392]) gives that

$$c_2(\widetilde{E}) - c_2(E) = l(Q) + l(R^1\pi_*\widetilde{E}). \tag{1.7}$$

We calculate $l(Q)$ and $l(R^1\pi_*\widetilde{E})$ for the bundles given by triples (E, j, u) and $(E, j, 0)$.

N. Buchdahl [2, Prop. 2.8] has already shown that these bounds are sharp in a more general setting by entirely different methods. However, our construction is entirely algebraic and can be used to calculate the charge $c_2(\widetilde{E}) - c_2(E)$ for any rank–2 holomorphic bundle on a blown–up surface. The method is described here for the case of vanishing first Chern class, but it is straightforward to further generalize it.

2. Calculation of Chern classes

As mentioned in the previous section, the charge $c_2(\widetilde{E}) - c_2(E)$ depends only of the restriction of $\widetilde{E}$ to the $(2j-2)$nd infinitesimal neighborhood of ℓ, which we denote by N_ℓ. We show that the upper bound occurs when $\widetilde{E}$ splits on N_ℓ, that is when $p = 0$ whereas the lower bound occurs for the extension,

$$0 \to \mathcal{O}(-j)|_{N_\ell} \to \widetilde{E}|_{N_\ell} \to \mathcal{O}(j)|_{N_\ell} \to 0$$

given by taking the polynomial p simply to be u, where $u = 0$ is the equation of the exceptional divisor. Note that the bundle $\widetilde{E}$ itself need not be an extension over $\widetilde{X}$.

Before proceeding to the results, let us first give a shortcut for the calculations. The computations of the numerical invariants $l(R^1\pi_*\widetilde{E})$ and $l(Q)$ involve calculating two inverse limits (see [5, p. 108]), which in general can become quite involved. However, in our case, some nice simplifications take place, which go as follows.

Lemma 2.1. *Suppose $\widetilde{E}$ is a bundle over $\widetilde{\mathbb{C}^2}$ given by transition matrix $T = \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$, where p is a polynomial of order $m \geq j$ in the variable u. Then $n \geq m$ implies that $\forall i$ $H^i(\ell_{n+1}, \widetilde{E}|\ell_{n+1})$ and $H^i(\ell_n, \widetilde{E}|\ell_n)$ have the same generators as $\mathbb{C}[[x,y]]$–modules.*

Proof. For $i > 1$, we have $H^i(\ell_n, \widetilde{E}|\ell_n) = 0$, which follows immediately from the fact that $\widetilde{\mathbb{C}^2}$ is covered by only to open sets. For $i = 1$, we consider the short exact sequence,

$$0 \to \widetilde{E} \otimes \mathcal{I}^n/\mathcal{I}^{n+1} \to \widetilde{E}|\ell_{n+1} \to \widetilde{E}|\ell_n \to 0. \tag{2.1}$$

Since $\mathcal{I}^n/\mathcal{I}^{n+1} \simeq \mathcal{O}(n)$ and $n \geq j$, we have that $H^i(\widetilde{E} \otimes \mathcal{I}^n/\mathcal{I}^{n+1}) = 0$ for $i \geq 1$; and the result follows from the long exact sequence in cohomology induced by (2.1).

For $i = 0$, first of all, it is clear that $M_n := H^0(\ell_n, \widetilde{E}|\ell_n) \subset M_{n+1} := H^0(\ell_{n+1}, \widetilde{E}|\ell_{n+1})$. If $\sigma \in M_{n+1}$ is a section of degree $n+1$ in u, then we want to show that σ belongs to the $\mathbb{C}[[x,y]]$–module generated by M_n. We use induction on n. Case $n = 0$ corresponds to the split bundle, for which the result is obvious. If $\sigma \in \Gamma(U \cap \ell_{n+1}, \widetilde{E}|\ell_{n+1})$, then σ has a power series expression of the form,

$$\sigma = \sum_{l=0}^{n+1} \sum_{k=0}^{\infty} \begin{pmatrix} a_{lk} \\ b_{lk} \end{pmatrix} z^k u^l.$$

If σ extends to a global section of $\widetilde{E}|\ell_{n+1}$ then the condition $T\sigma \in \Gamma(V \cap \ell_{n+1}, \widetilde{E}|\ell_{n+1})$ must hold, that is,

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \sum_{l=0}^{n+1} \sum_{k=0}^{\infty} \begin{pmatrix} a_{lk} \\ b_{lk} \end{pmatrix} z^k u^l \tag{2.2}$$

must be holomorphic in the variables $\xi = z^{-1}$ and $v = z\,u$. Since x acts by multiplication by u, the equality

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} 0 \\ z^k u^{n+1} \end{pmatrix} = u \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} 0 \\ z^k u^n \end{pmatrix}$$

shows that $\alpha := \begin{pmatrix} 0 \\ z^k u^{n+1} \end{pmatrix} \in M_{n+1}$ if and only if $\begin{pmatrix} 0 \\ z^k u^n \end{pmatrix} \in M_n$; in which case, α belongs to the $\mathbb{C}[[x,y]]$–module generated by M_n. Hence, it suffices to verify whether $\sigma - \alpha \in M_n$. Using the power series expression of $\sigma - \alpha$ we restate the condition $T(\sigma - \alpha) \in \Gamma(V \cap \ell_{n+1}, \widetilde{E}|\ell_{n+1})$. Thus, we obtain conditions in the coefficients of u^m, which by induction hypothesis are satisfied for $0 \leq m \leq n$. We are then left with one single extra condition; namely, that

$$z^j a_{n+1,k} z^k u^{n+1} + p \sum_{l=0}^{n} \sum_{k=0}^{\infty} b_{n+1,k} z^k u^l \tag{2.3}$$

be holomorphic in z^{-1} and zu. But because p has order $m \leq n$ in u there are no elements of the form $p\, b_{0,k+1}$ which are of order $n+1$ in u. Hence, we can factor u in (2.3) and, equivalently, demand that

$$z^j a_{n+1,k} z^k u^n + p \sum_{l=0}^{n} \sum_{k=0}^{\infty} b_{l+1,k} z^k u^l \tag{2.4}$$

be holomorphic in z^{-1} and zu. Thus, we arrived at the same condition that holds true in M_n (up to renaming the coefficients), and hence is satisfied by inductive hypothesis. It follows that $\sigma - \alpha$, and hence σ, belongs to the $\mathbb{C}[[x,y]]$–module generated by M_n. □

Lemma 2.2. *Set* $M = H^0(N_\ell, \widetilde{E}|_{N_\ell})$, *and let* $\rho\colon M \hookrightarrow M^{\vee\vee}$ *be the natural inclusion of* M *into its bidual. Then* $l(Q) =$*dim coker* ρ.

Proof. Since the length of Q equals the dimension of ${Q_x}^\wedge$ as a $\mathbb{C}$–vector space, and since Q is defined by the sequence (1.6), we need to study the map $(\pi_*\widetilde{E}_x)^\wedge \to {E_x}^\wedge$ and compute the dimension of the cokernel as a $\mathbb{C}$–vector space. But as ${E_x}^\wedge = (\pi_*\widetilde{E}_x)^{\wedge\vee\vee}$, we need to compute the $\mathbb{C}[[x,y]]$–module structure on $W := (\pi_*\widetilde{E}_x)^\wedge$ and to study the natural map $W \hookrightarrow W^{\vee\vee}$. By the Theorem on Formal Functions (see [4, p. 277])

$$W \simeq \lim_{\longleftarrow} H^0(\ell_n, \widetilde{E}|\ell_n)$$

as $\mathbb{C}[[x,y]]$–modules, where ℓ_n is the n-th infinitesimal neighborhood of ℓ. Since $N_\ell := \ell_{2j-2}$, and since p has degree $2j-2$ in u, by Lemma 2.1, if $n \geq 2j-2$, then

$$H^0(\ell_n, \widetilde{E}|\ell_n) \simeq H^0(N_\ell, \widetilde{E}|N_\ell)$$

as $\mathbb{C}[[x,y]]$–modules. Therefore, by Thm. 9.3 on [4, p. 193], the inverse limit stabilizes at $2j-2$, thus giving $W \simeq H^0(N_\ell, \widetilde{E}|N_\ell) = M$. □

Lemma 2.3. $l(R^1\pi_*\widetilde{E}) = dim_{\mathbb{C}}\ H^1(N_\ell, \widetilde{E}|_{N_\ell})$.

Proof. This proof is similar to that of Lemma 2.2. We use the Theorem on Formal Functions [4, p. 277], which gives

$$R^1\pi_*\widetilde{E} = \lim_{\longleftarrow}\ H^1(\ell_n, \widetilde{E}|_{\ell_n}).$$

We see from (1.3) that the polynomial p which determines the extension $\widetilde{E}$ has degree $2j-2$ in u, and applying Lemma 2.1 we see that the inverse limit stabilizes at $n = 2j-2$. It then follows from [4, Thm. 9.3] that $R^1\pi_*\widetilde{E} = H^1(\ell_{2j-2}, \widetilde{E}|_{\ell_{2j-2}})$. Since $N_\ell := 2j-2$ we conclude that $l(R^1\pi_*\widetilde{E}) = dim_{\mathbb{C}}\ H^1(N_\ell, \widetilde{E}|_{N_\ell})$. □

Note that these simplifications are only possible because we have algebraic bundles whose extension class is explicitly given in (1.4) (see also [2]). For holomorphic bundles that are not algebraic, more work is required to compute the inverse limits. In [1, p. 3–4] we calculate several instances of $l(Q)$ and $l(R^1\pi_*\widetilde{E})$ for extensions of $\mathcal{O}(j)$ by $\mathcal{O}(-j)$ for $j = 2, 3$. We now prove sharpness of the bounds.

3. The upper bound occurs for $p = 0$

We show that the upper bound for the charge occurs for the split extension, which by (1.5) is denoted $\widetilde{E} := (E, j, 0)$.

Theorem 3.1. *If* $\widetilde{E} := (E, j, 0)$*, then* $c_2(\widetilde{E}) - c_2(E) = j^2$.

Proof. Use lemmas 3.2 and 3.3 together with equality (1.7). ☐

Lemma 3.2. *If* $\widetilde{E} := (E, j, 0)$*, then* $l(Q) = j(j+1)/2$.

Proof. According to Lemma 2.2, $l(Q) = \dim \operatorname{coker} \rho$ is the dimension of the cokernel of the natural inclusion $\rho: M \hookrightarrow M^{\vee\vee}$ where $M = H^0(N_\ell, \widetilde{E}|_{N_\ell})$.

We find that $M = \mathbb{C}[[x, y]] < \alpha, \beta_0, \beta_1, ..., \beta_j > /R$ is generated by

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \beta_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \beta_1 = \begin{pmatrix} 0 \\ z \end{pmatrix},\ \cdots,\ \beta_j = \begin{pmatrix} 0 \\ z^j \end{pmatrix}$$

satisfying the set of relations

$$R := \begin{cases} x\beta_1 - y\beta_0 = 0,\ x\beta_2 - y\beta_1 = 0 \cdots, x\beta_j - y\beta_{j-1} = 0, \\ x^2\beta_2 - y^2\beta_0 = 0, \cdots, x^2\beta_j - y^2\beta_{j-2} = 0 \\ \vdots \\ x^j\beta_j - y^j\beta_0 = 0 \end{cases}.$$

All together there are $j(j+1)/2$ independent relations. Then, writing the generators of $M^\vee$ and $M^{\vee\vee}$, we see that $\operatorname{coker}(M \hookrightarrow M^{\vee\vee})$ is a $j(j+1)/2$ dimensional vector space over $\mathbb{C}$. Hence $l(Q) = j(j+1)/2$. ☐

Lemma 3.3. *If* $\widetilde{E} := (E, j, 0)$*, then* $l(R^1\pi_*\widetilde{E}) = j(j-1)/2$.

Proof. According to Lemma 2.3, we have $l(R^1\pi_*\widetilde{E}) = dim_{\mathbb{C}}\ H^1(N_\ell, \widetilde{E}|_{N_\ell})$. It is simple to calculate $H^1(N_\ell, \widetilde{E}|_{N_\ell})$ directly from the explicit form of transition matrix for $\widetilde{E}|_{N_\ell}$ and we find that $l(R^1\pi_*\widetilde{E}) = j(j-1)/2$. ☐

Remark: We can also proof Theorem 3.1 in a simpler way, by explicitly constructing a generic section of $\widetilde{E}$ and counting its zeros. However, the method of finding a generic section is manageable only when $\widetilde{E}$ splits near ℓ, whereas our application of the Theorem on Formal Functions is completely general and works whether $\widetilde{E}$ splits or not.

4. The lower bound occurs for $p = u$

We show that the upper bound for the charge occurs when we take the extension class p simply to be u, which by (1.5) is denoted $\widetilde{E} := (E, j, u)$.

Theorem 4.1. *If $\widetilde{E} := (E, j, u)$, then $c_2(\widetilde{E}) - c_2(E) = j$.*

Proof. Use lemmas 4.2 and 4.3 together with equality (1.7). □

Lemma 4.2. *If $\widetilde{E} := (E, j, u)$, then $l(Q) = 1$.*

Proof. The bundle $\widetilde{E}$ is given over N_ℓ by the transition matrix $\begin{pmatrix} z^j & u \\ 0 & z^{-j} \end{pmatrix}$ as in (1.2). Here calculations are similar to the ones in the proof of Lemma 3.2. We set $M := (\pi_* \widetilde{E}_x)^\wedge$ and study the natural map $\rho\colon M \hookrightarrow M^{\vee\vee}$ of $\mathcal{O}_x^\wedge$–modules. According to Lemma 2.2, there is an isomorphism $M \simeq H^0(N_\ell, \widetilde{E}|_{N_\ell})$. We find that $M = \mathbb{C}[[x,y]] < \beta_0, \beta_1, \beta_j, \alpha > /R$ where $\beta_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \beta_1 = \begin{pmatrix} 0 \\ z \end{pmatrix}, \beta_j = \begin{pmatrix} -u \\ z^j \end{pmatrix}, \alpha = \begin{pmatrix} u^j \\ 0 \end{pmatrix}$ and R is the set of relations

$$\begin{cases} x\,\beta_1 - y\,\beta_0 = 0 \\ \alpha + x^{j-1}\beta_j - y^{j-1}\beta_1 = 0 \end{cases}.$$

Using the second relation, we eliminate α from the set of generators, and get a simpler presentation $M \simeq \mathbb{C}[[x,y]] < \beta_0, \beta_1, \beta_j > /R'$ where R' consists of the single relation $x\,\beta_1 - y\,\beta_0 = 0$. It is now a matter of simple algebra to find that $M^\vee =< a, b >$ is free on two generators, where

$$a = \begin{cases} \beta_0 \mapsto x \\ \beta_1 \mapsto y \\ \beta_j \mapsto 0 \end{cases}, \quad b = \begin{cases} \beta_0 \mapsto 0 \\ \beta_1 \mapsto 0 \\ \beta_j \mapsto 1 \end{cases}.$$

Then $M^{\vee\vee} =< a^*, b^* >$ is generated by the dual basis, namely

$$a^* = \begin{cases} a \mapsto 1 \\ b \mapsto 0 \end{cases}, \quad b^* = \begin{cases} a \mapsto 0 \\ b \mapsto 1 \end{cases}.$$

Hence im $\rho =< x\,a^*, y\,a^*, b^* >$, and *coker* $\rho =< \overline{a^*} >$. So $l(Q) = \dim \operatorname{coker}\rho = 1$. □

Lemma 4.3. *If* $\widetilde{E} := (E, j, u)$, *then* $l(R^1\pi_*\widetilde{E}) = j - 1$.

Proof. We claim that $H^1(N_\ell, \widetilde{E}|_{N_\ell})$ is generated as a $\mathbb{C}$–vector space by the 1–cocycles $\binom{z^k}{0}$ for $-j+1 \le k \le -1$. Hence by Lemma 2.3

$$l(R^1\pi_*\widetilde{E}) = \dim \varprojlim H^1(\ell_n, \widetilde{E}|_{\ell_n}) = dim\ H^1(N_\ell, \widetilde{E}|_{N_\ell}) = j - 1.$$

In fact, if T is the transition matrix for $\widetilde{E}|_{N_\ell}$, then the two equations

$$B = \sum_{i=0}^{\infty} \sum_{k=-\infty}^{\infty} \begin{pmatrix} 0 \\ v_{ik} \end{pmatrix} = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \begin{pmatrix} 0 \\ v_{ik} \end{pmatrix} + T^{-1} \sum_{i=0}^{\infty} \sum_{k=-\infty}^{-1} \begin{pmatrix} v_{ik}\, u \\ v_{ik}\, z^{-j} \end{pmatrix}$$

where $v_{ik} := b_{ik} z^k u^i$, show that B is a coboundary, since the first term on the right hand side is holomorphic in U and the last term is holomorphic on V. As a consequence every 1–cocycle has a representative of the form $\alpha = \sum_{i=0}^{\infty} \sum_{k=-\infty}^{\infty} \binom{a_{ik} z^k u^i}{0}$. Similarly, the equality

$$A = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \begin{pmatrix} w_{ik} \\ 0 \end{pmatrix} + T^{-1} \sum_{i=0}^{\infty} \sum_{k=-\infty}^{-1} \begin{pmatrix} w_{ik} \\ 0 \end{pmatrix}$$

where $w_{ik} := a_{ik} z^k u^i$ shows that A is a coboundary. Hence, $\alpha \sim \alpha - A = \sum_{k=-j+1}^{-1} \binom{w_{0k}}{0}$. Therefore, the nonvanishing cohomology classes correspond to the terms $\binom{w_{0k}}{0}$ for $-j+1 \le k \le -1$, thus giving the $j-1$ generators of $H^1(N_\ell, \widetilde{E}|_{N_\ell})$. □

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