Vector bundles near negative curves: moduli and local Euler characteristic

E. Ballico E. Gasparim T. Köppe

Abstract

We study moduli of vector bundles on a two-dimensional neighbourhood Z_k of an irreducible curve $\ell \cong \mathbb{P}^1$ with $\ell^2 = -k$ and give an explicit construction of their moduli stacks. For the case of instanton bundles, we stratify the stacks and construct moduli spaces. We give sharp bounds for the local holomorphic Euler characteristic of bundles on Z_k and prove existence of families of bundles with prescribed numerical invariants. Our numerical calculations are performed using a *Macaulay 2* algorithm, which is available for download at `http://www.maths.ed.ac.uk/~s0571100/Instanton/`.

1 Introduction

We study moduli spaces of rank-2 bundles on a two-dimensional neighbourhood of an irreducible curve $\ell \cong \mathbb{P}^1$ with negative self-intersection $\ell^2 = -k \neq 0$. We are interested in the behaviour of bundles over a small analytic neighbourhood of ℓ inside a smooth surface Z_k, and in coherent sheaves near the singular point of the surface X_k obtained from Z_k by contracting the curve ℓ. For this "local" problem of bundles near ℓ it is enough to focus on vector bundles over the total space of $\mathcal{O}_{\mathbb{P}^1}(-k)$. Hence we take $Z_k := \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k)\big)$, where $\ell \subset Z_k$ is the zero section. We write $\pi\colon Z_k \to X_k$ for the map that contracts ℓ to a point. We give an explicit construction of the moduli stack of rank-2 bundles on Z_k.

A bundle E over Z_k has *splitting type* $(j_1, \dots, j_r)$ if $E|_\ell \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(j_i)$ with $j_1 \leq \cdots \leq j_r$. For the moduli problem we concentrate on the case of rank-2 bundles E with vanishing first Chern class; in this case the splitting type of E must be $(-j, j)$ for some $j \geq 0$, and for short we say that E has splitting type j. We then define

$$\mathcal{M}_j(k) = \big\{E \to Z_k : E|_\ell \cong \mathcal{O}(j) \oplus \mathcal{O}(-j)\big\} \big/\!\sim$$

for the moduli (stack) of bundles over Z_k of splitting type j. We prove:

Theorem 4.11. *For $j \geq k$, $\mathcal{M}_j(k)$ has an open, dense subspace homeomorphic to a complex projective space $\mathbb{P}^{2j-k-2}$ minus a closed subvariety of codimension at least $k+1$.*

The moduli $\mathcal{M}_j(k)$ contains only one point if $2j - 2 < k$, and it is non-Hausdorff for $2j - 2 \geq k$. The cases when $j = nk$, that is, when the splitting type is a multiple of $k = -\ell^2$, are of special interest for applications to physics, because they correspond to instantons. We cite:

Theorem ([GKM, Corollary 5.5]). *An $\mathfrak{sl}(2,\mathbb{C})$-bundle over Z_k represents an instanton if and only if its splitting type is a multiple of k.*

We give a stratification of the instanton moduli stacks, that is of $\mathcal{M}_j(k)$ for the case $j = nk$, dividing it into Hausdorff components (in the analytic topology). For such a stratification, we need numerical invariants, which we now define: Given a vector bundle E over Z_k, we define the Artinian sheaf Q_E on X_k by the exact sequence

$$0 \longrightarrow \pi_* E \longrightarrow (\pi_* E)^{\vee\vee} \longrightarrow Q_E \longrightarrow 0 \ . \tag{1.1}$$

E has two independent invariants, called *height* and the *width*. The terminology comes from instantons, cf. [Ga4].

Definition 1.1. We define the *height* and *width* of E by

$$\mathbf{h}_k(E) := \operatorname{length} R^1\pi_* E \quad \text{and} \quad \mathbf{w}_k(E) := \operatorname{length} Q_E \ .$$

We show:

Theorem 4.15. *If $j = nk$ for some $n \in \mathbb{N}$, then the pair $(\mathbf{h}_k, \mathbf{w}_k)$ stratifies instanton moduli stacks $\mathcal{M}_j(k)$ into Hausdorff components.*

Remark 1.2. Note that the invariants are defined for any bundle (or sheaf) over Z_k but we only claim the stratification result for instantons, that is, for the case when $j = nk$ is a multiple of k. In fact, this is a necessary condition, and $\mathcal{M}_3(2)$ is already an example that justifies the necessity of the "instanton" condition (see Example 4.18), and application of the embedding theorem 4.12 provides infinitely many such cases.

One could also decompose $\mathcal{M}_j(k)$ by the local holomorphic Euler characteristic $\chi(\ell, E)$ (see Definition 1.3). However, $\mathcal{M}_j(k)$ has non-Hausdorff subspaces with fixed $\chi(\ell, E)$, of which the simplest example is $\mathcal{M}_3(1)$ (see Example 4.17). See [BG1] for the case of an exceptional curve, i.e. $k = 1$. We prove a simple general formula for the height:

Theorem 2.7. *Let E be the holomorphic, rank-2, non-split vector bundle of splitting type j which is represented in canonical form (2.3) by p, and let $m > 0$ be the smallest exponent of u appearing in p. With $\mu = \min\big(m, \lfloor \frac{j-2}{k} \rfloor\big)$, we have*

$$l(R^1\pi_* E) \geq \mu \left(j - 1 - k\, \frac{\mu - 1}{2} \right) \ ,$$

and equality holds if p is holomorphic on Z_k.

We then calculate explicitly sharp bounds for the invariants $(\mathbf{h}_k, \mathbf{w}_k)$:

Theorems 2.8 and 2.16. *Let E be a holomorphic rank-2 vector bundle over Z_k with $c_1 = 0$ and splitting type $j > 0$. Let $n_1 = \big\lfloor \frac{j-2}{k} \big\rfloor$ and $n_2 = \big\lfloor \frac{j}{k} \big\rfloor$. Then the following bounds are sharp:*

$$j - 1 \leq \mathbf{h}_k(E) \leq (j-1)(n_1+1) - k n_1 (n_1+1)/2 \ ,$$
$$0 \leq \mathbf{w}_k(E) \leq (j+1) n_2 - k n_2 (n_2+1)/2 \ ,$$

and $\mathbf{w}_1(E) \geq 1$.

Definition 1.3 ([Bl, Def. 3.9])**.** Let $\sigma(\widetilde{X},\ell)\to(X,x)$ be a resolution of an isolated quotient singularity. Let $\widetilde{\mathcal{F}}$ be a reflexive sheaf of rank n on $\widetilde{X}$, and set $\mathcal{F}:=(\sigma_*\widetilde{\mathcal{F}})^{\vee\vee}$; notice that there is a natural injection $\sigma_*\widetilde{\mathcal{F}}\hookrightarrow\mathcal{F}$. Then the *local holomorphic Euler characteristic* of $\widetilde{\mathcal{F}}$ is

$$\chi\big(x,\widetilde{\mathcal{F}}\big):=\chi\big(\ell,\widetilde{\mathcal{F}}\big):=h^0\big(X;\mathcal{F}/\sigma_*\widetilde{\mathcal{F}}\big)+\sum_{i=1}^{n}(-1)^{i-1}h^0\big(X;R^i\sigma_*\widetilde{\mathcal{F}}\big)\ .$$

Corollary 2.18. *Let E be a rank-2 bundle over Z_k of splitting type $j>0$ and let $j=nk+b$ such that $0\leq b<k$. The following are sharp bounds for the local holomorphic Euler characteristic of E:*

$$j-1\leq\chi(\ell,E)\leq\begin{cases}n^2k+2nb+b-1 & \text{if } k\geq 2 \text{ and } 1\leq b<k\ ,\\ n^2k & \text{if } k\geq 2 \text{ and } b=0\ ,\end{cases}$$

and

$$j\leq\chi(\ell,E)\leq j^2 \text{ for } k=1\ .$$

Next, we consider the question of existence of vector bundles. We recall the concept of an *admissible sequence* (Definition 3.1) and prove the following existence result:

Theorem 3.2. *Fix an admissible sequence $\big\{a(i,l)\big\}_{i=1}^t$ and let E and F be rank-r vector bundles on $\widehat{\ell}$ with $\big\{a(i,l)\big\}_{i=1}^t$ as an associated admissible sequence. Then there exists a flat family $\big\{E_s\big\}_{s\in T}$ of rank-r vector bundles on $\widehat{\ell}$ parametrised by an integral variety T and $s_0,s_1\in T$ with $E_{s_0}\cong E$ and $E_{s_1}\cong F$ such that E_s has $\big\{a(i,l)\big\}_{i=1}^t$ as admissible sequence for every $s\in T$.*

In Section 2 we calculate numerical invariants for bundles near negative curves. In Section 3 we describe the method of *balancing* bundles and show the existence of families of bundles with prescribed numerical invariants. In Section 4 we study moduli of rank-2 bundles on Z_k.

Our calculations were often performed on a computer using an implementation of the height and width computations as described in [GKM] written with the computer algebra software *Macaulay 2* [M2]. The program can be downloaded from `http://www.maths.ed.ac.uk/ ~s0571100/Instanton/`.

The computer program was essential for discovering the results of Sections 2 and 4; algebraic calculations are straightforward but too laborious to carry out by hand. The original program was developed by I. Swanson in [GS] for bundles on Z_1, where the contraction $\pi\colon Z_1\to X_1$ has the lifting map $\widetilde{\pi}\colon\mathcal{O}_{X_1}\to\pi_*\mathcal{O}_{Z_1}$ given in coordinates by $x\mapsto u, y\mapsto zu$. We generalised the computation to the case $Z_k\to X_k$, where X_k has the coordinate ring $\mathbb{C}[x_0,x_1,\ldots,x_k]\big/\{x_ix_{i+h}-x_{i+1}x_{i+h-1}\}$ for $0\leq i\leq i+h\leq k$ and the lifting map is $x_i\mapsto z^iu$, but the algorithms for the computation of the module of sections and the width are otherwise essentially the same.

Acknowledgments. We would like to thank the editor for having done such a wonderful job, and we thank the referee for several helpful suggestions.

2 Bounds

Let Z_k be the total space of $\mathcal{O}_{\mathbb{P}^1}(-k)$ and $\ell \cong \mathbb{P}^1$ the zero section, defined by the ideal sheaf $\mathcal{I}_\ell$, so that $\ell^2 = -k$. We write

$$\ell_N = \big(\ell,\ \mathcal{O}_{Z_k}/\mathcal{I}_\ell^{N+1}|_\ell\big)$$

for the N^{th} infinitesimal neighbourhood of ℓ, $\widehat{\ell} = \varprojlim \ell_N$ for the formal neighbourhood of ℓ in Z_k, and $\mathcal{O}(j)$ for the line bundle on Z_k or on $\widehat{\ell}$ that restricts to $\mathcal{O}_{\mathbb{P}^1}(j)$ on ℓ. A vector bundle E has *splitting type* $(j_1, \ldots, j_r)$ if $E|_\ell \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(j_i)$ with $j_1 \geq \cdots \geq j_r$. A result of Griffiths [Gr] implies that E splits on $\widehat{\ell}$ if $j_1 - j_r \leq k+1$.

Lemma 2.1. *Let Z be a smooth surface containing a curve $\ell \cong \mathbb{P}^1$ with $\ell^2 = -k$. Let E be a rank-r vector bundle on Z of splitting type $j_1 \geq j_2 \geq \cdots \geq j_r$, and assume that $j_1 - j_r \leq k+1$. Then E splits on the formal neighbourhood $\widehat{\ell}$ of ℓ, that is, $E|_{\widehat{\ell}} \cong \bigoplus_{i-1}^r \mathcal{O}_{\widehat{\ell}}(j_i)$.*

Proof. This follows from [Gr, Propositions 1.1 and 1.4]: The point is that $\mathscr{H}om_{\mathcal{O}_{\widehat{\ell}}}(E,E)|_\ell$ is a direct sum of line bundles of degree $\geq -(j_1 - j_r)$, so the first-order infinitesimal extensions $H^1\big(\widehat{\ell}; \mathscr{H}om_{\mathcal{O}_{\widehat{\ell}}}(E,E) \otimes \mathcal{O}(-\ell)\big)$ vanish. □

As an aside, we note a relation with the McKay correspondence (see e.g. [Re]), which relates the (co)homology of a resolution $Y \to \mathbb{C}^2/G$ to the irreducible representations of a finite group G of automorphisms: It is known that a reflexive sheaf on a surface quotient singularity $\mathbb{C}^2/G$ is a direct sum of the tautological sheaves obtained from the irreducible representations of G. For cyclic quotient singularities $\frac{1}{r}(1,a)$ these are just the eigensheaves $\mathcal{O}(i)$ of the group action, such that $\pi_*\mathcal{O} = \bigoplus_{i=0}^{r-1} \mathcal{O}(i)$.

We now study the holomorphic invariants $\mathbf{h}_k(E) = l(R^1\pi_*E)$ and $\mathbf{w}_k(E) = l(Q_E)$ as in Definition 1.1.

Remark 2.2. In principle, we need the Theorem on Formal Functions [Ha, p. 276], to calculate $\mathbf{w}_k$ and $\mathbf{h}_k$:

$$\mathbf{w}_k(E) = \dim_{\mathbb{C}} \operatorname{coker}\big(\rho\colon M \hookrightarrow M^{\vee\vee}\big)\ , \quad \text{where } M := \varprojlim H^0\big(\ell_n; E|_{\ell_n}\big)\ , \text{ and}$$

$$\mathbf{h}_k(E) = \dim_{\mathbb{C}} \varprojlim H^1\big(\ell_n; E|_{\ell_n}\big)\ .$$

However, since holomorphic bundles on Z_k are algebraic, the limit stabilises at a finite order and it is enough to compute the cohomology on a fixed infinitesimal neighbourhood ℓ_N, where N is not too small (see [Ga3, Lemmas 2.1–2.3]) — we restate these results in a slightly generalised form in Lemma 2.5 below.

It follows from $H^*\big(\bigoplus_{i=1}^N \mathcal{F}_i\big) = \bigoplus_{i=1}^N H^*\big(\mathcal{F}_i\big)$ for coherent sheaves $\mathcal{F}_i$ that heights and widths behave additively for bundles that split on the formal neighbourhood $\widehat{\ell}$:

Proposition 2.3. *Let E be a holomorphic vector bundle on Z_k. If $E|_{\widehat{\ell}} \cong \bigoplus_{i=1}^r \mathcal{O}(j_i)$, then*

$$\mathbf{w}_k(E) = \sum_{i=1}^r \mathbf{w}_k\big(\mathcal{O}(j_i)\big) \quad \textit{and} \quad \mathbf{h}_k(E) = \sum_{i=1}^r \mathbf{h}_k\big(\mathcal{O}(j_i)\big)\ . \qquad \square$$

Corollary 2.4. *Let E be a holomorphic rank-r vector bundle over Z_k such that $E|_\ell \cong \bigoplus_{i=1}^r \mathcal{O}_\ell(j_i)$, with $j_1 \geq j_2 \geq \cdots \geq j_r$ and $j_r - j_1 \geq -k-1$. Then*

$$\mathbf{w}_k(E) = \sum_{i=1}^r \mathbf{w}_k\big(\mathcal{O}(j_i)\big) \quad \text{and} \quad \mathbf{h}_k(E) = \sum_{i=1}^r \mathbf{h}_k\big(\mathcal{O}(j_i)\big) \ .$$

Proof. By Lemma 2.1, $E|_{\widehat{\ell}} \cong \bigoplus_{i=1}^r \mathcal{O}_{\widehat{\ell}}(j_i)$. The result follows from Proposition 2.3 (and hence E has the same invariants as the split bundle). □

We fix once and for all coordinate charts on Z_k, to which we will refer as *canonical coordinates*,

$$U = \mathbb{C}^2_{z,u} = \big\{z, u\big\} \qquad \text{and} \qquad V = \mathbb{C}^2_{\zeta,v} = \big\{\zeta, v\big\} \ , \tag{2.1}$$

glued by $\zeta = z^{-1}$ and $v = z^k u$. In these charts the bundle $\mathcal{O}(j)$ has the transition matrix $\big(z^{-j}\big)$.

Bundles with vanishing first Chern class are of special interest for applications to physics. For example, the bundles with splitting type nk correspond to framed instantons under a local version of the Kobayashi–Hitchin correspondence (see [GKM] and [LT]). For the case $c_1 = 0$, we also calculate the lower bounds for the numerical invariants $\mathbf{h}$ and $\mathbf{w}$. The second author proved in [Ga1] that holomorphic bundles on Z_k are algebraic extensions of line bundles. By [Ga1, Theorem 3.3], a bundle E that is an extension

$$0 \longrightarrow \mathcal{O}(j_1) \longrightarrow E \longrightarrow \mathcal{O}(j_2) \longrightarrow 0 \tag{2.2}$$

(with $j_1 \leq j_2$) has transition matrix $T = \left(\begin{smallmatrix} z^{-j_1} & p(z,u) \\ 0 & z^{-j_2}\end{smallmatrix}\right)$ in canonical coordinates (2.1), where the extension class

$$\big[p\big] \in \operatorname{Ext}^1_{\mathcal{O}_{Z_k}}\big(\mathcal{O}(j_2),\ \mathcal{O}(j_1)\big)$$

may be represented by a *canonical form*

$$p(z,u) = \sum_{r=1}^{\lfloor (j_2-j_1-2)/k \rfloor} \ \sum_{s=kr+j_1+1}^{j_2-1} p_{rs}\, z^s u^r \ . \tag{2.3}$$

Lemma 2.5 (Computation on finite neighbourhoods)**.** *Let E be a rank-2 bundle of splitting type j on Z_k and set $N := \lfloor (2j-2)/k \rfloor$. Then the height and width of E are determined already on ℓ_N in the following sense:*

- $\mathbf{h}_k(E) = \dim_{\mathbb{C}} H^1\big(\ell_N; E|_{\ell_N}\big)$.
- *Set $M = H^0\big(\ell_N; E|_{\ell_N}\big)$ and let $\rho\colon M \hookrightarrow M^{\vee\vee}$ be the evaluation map. Then $\mathbf{w}_k(E) = \dim_{\mathbb{C}} \operatorname{coker} \rho$.*

Proof. The crucial fact is that the extension class is given by a polynomial p whose degree in u is at most $N := \lfloor (2j-2)/k \rfloor$. Now it was shown in [Ga3, Lemmas 2.1] for the case $k = 1$, but readily generalised to all $k \geq 1$, that for $n \geq N$, the modules $H^i\big(\ell_n; E|_{\ell_n}\big)$ and $H^i\big(\ell_N; E|_{\ell_N}\big)$ have the same module structure as $\mathcal{O}_0^\wedge$-modules for all $i \geq 0$, where $\pi(\ell) = 0 \in X_k$. Hence by [Ha, p. 193], the inverse limits from Remark 2.2 stabilise at $n = N$, which proves the Lemma. □

2.1 Heights

We begin with the computation of the height of a line bundle:

Lemma 2.6. *Assume $j \geq 0$ and let $n_1 = \left\lfloor \frac{j-2}{k} \right\rfloor$. Then*

$$\mathbf{h}_k\big(\mathcal{O}(-j)\big) = \begin{cases} (j-1)(n_1+1) - kn_1(n_1+1)/2 & \text{if } j \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. $R^1\pi_*\mathcal{O}(j) = \varprojlim H^1\big(\mathcal{O}_{\ell_n}(j)\big)$, with surjective restriction maps. The result comes from the exact sequences

$$0 \longrightarrow H^1\big(\mathcal{O}_\ell(-n\ell) \otimes \mathcal{O}(j)\big) \longrightarrow H^1\big(\mathcal{O}_{\ell_{n+1}}(j)\big) \longrightarrow H^1\big(\mathcal{O}_{\ell_n}(j)\big) \longrightarrow 0$$

together with

$$H^1\big(\mathcal{O}_\ell(-n\ell) \otimes \mathcal{O}(j)\big) = H^1\big(\mathbb{P}^1; \mathcal{O}(j+nk)\big) \ ,$$

which gives $\mathbf{h}_k\big(\mathcal{O}(-j)\big) = \sum_{n=0}^{\infty}(j-1-nk)^+$, where $^+$ means the sum of positive terms only. $\square$

This result together with Proposition 2.3 allows us to compute the heights of split bundles. For non-split bundles with $c_1 = 0$ and splitting type $j := j_2 = -j_1$, we use the following:

Theorem 2.7. *Let E be the non-split bundle of splitting type j represented in canonical form (2.3) by p, and let $m > 0$ be the smallest exponent of u appearing*[1] *in p. With $\mu = \min\big(m, \lfloor \frac{j-2}{k} \rfloor\big)$, we have*

$$l(R^1\pi_*E) \geq \mu \left(j - 1 - k\, \frac{\mu - 1}{2} \right) \ ,$$

and equality holds if p is holomorphic on Z_k.

Proof. By Lemma 2.9 below, a cocycle in $H^1(E)$ has the canonical representation on the U-chart

$$\sum_{r=0}^{\lfloor \frac{j-2}{k} \rfloor} \sum_{s=kr-j+1}^{-1} \binom{a_{rs}}{0} z^s u^r \ .$$

In this representation, every monomial term $(a_{rs}z^su^r, 0)$ with $r < m$ represents a non-trivial cocycle by Lemma 2.10. Lastly, if p is holomorphic in Z_k, it has only terms $p_{rs}z^su^r$ with $0 \leq s \leq kr$, and then all terms with $r \geq m$ are coboundaries by Lemma 2.11. $\square$

Putting the split and the non-split cases together, we get sharp bounds on the heights:

Theorem 2.8. *Let E be a rank-2 bundle over Z_k of splitting type $j > 0$. Set $n_1 = \left\lfloor \frac{j-2}{k} \right\rfloor$. The following bounds are sharp:*

$$j - 1 \leq \mathbf{h}_k(E) \leq (j-1)(n_1+1) - k(n_1+1)n_1/2 \ .$$

[1] A rank-2 bundle whose extension class is a polynomial not divisible by u is in fact not of splitting type j, but of a lower splitting type. See Remark 2.13.

Proof. The upper bound is attained by the split bundle by Proposition 2.12. In this case, apply Lemma 2.6 and Proposition 2.3.

For the lower bound, note that the expression in Theorem 2.7 is always less than in or equal to the split case. The global minimum is $j-1$, attained with $\mu = 1$ in Theorem 2.7, attained by $p(z,u) = zu$. □

We finish this subsection by proving the details of Theorem 2.7. Recall once and for all that in the charts $Z_k = U \cup V$ given by (2.1), a function is holomorphic on U if it is holomorphic in $\{z, u\}$, and on V if it is holomorphic in $\{z^{-1}, z^k u\}$.

Lemma 2.9. *Every* 1*-cocycle in* $H^1(E)$ *has a representative of the form*

$$\sum_{r=0}^{\lfloor \frac{j-2}{k} \rfloor} \sum_{s=kr-j+1}^{-1} \begin{pmatrix} a_{rs} \\ 0 \end{pmatrix} z^s u^r \ ,$$

with $a_{rs} \in \mathbb{C}$*. In particular, every* 1*-cochain represented by* $\left(\begin{smallmatrix} a_{rs} \\ 0 \end{smallmatrix}\right) z^s u^r$ *with* $r, s \geq 0$ *is a coboundary.*

Proof. Let σ be a 1-cocycle and let $\sim$ denote cohomological equivalence. A power series representative for a 1-cochain has the form

$$\sigma = \sum_{r=0}^{\infty} \sum_{s=-\infty}^{\infty} \begin{pmatrix} a_{rs} \\ b_{rs} \end{pmatrix} z^s u^r \ ,$$

with $a_{rs}, b_{rs} \in \mathbb{C}$. The 1-cochain $s_1 = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \left(\begin{smallmatrix} a_{rs} \\ b_{rs} \end{smallmatrix}\right) z^s u^r$ is holomorphic on U, hence represents a coboundary. Consequently

$$\sigma \sim \sigma - s_1 = \sum_{r=0}^{\infty} \sum_{s=-\infty}^{-1} \begin{pmatrix} a_{rs} \\ b_{rs} \end{pmatrix} z^s u^r \ .$$

Now let $T = \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$ be the transition function of E, so that after a change of coordinates,

$$T\sigma = \sum_{r=0}^{\infty} \sum_{s=-\infty}^{-1} \begin{pmatrix} z^j a_{rs} + p\, b_{rs} \\ z^{-j} b_{rs} \end{pmatrix} z^s u^r \ .$$

However, given that $s_2 = \sum_{r=0}^{\infty} \sum_{s=-\infty}^{-1} \left(\begin{smallmatrix} 0 \\ z^{-j} b_{rs} \end{smallmatrix}\right) z^s u^r$ is holomorphic on V,

$$T\sigma \sim T\sigma - s_2 = \sum_{r=0}^{\infty} \sum_{s=-\infty}^{-1} \begin{pmatrix} z^j a_{rs} + p\, b_{rs} \\ 0 \end{pmatrix} z^s u^r \ ,$$

and going back to the U-coordinate chart,

$$\sigma = T^{-1} T\sigma \sim \sum_{r=0}^{\infty} \sum_{s=-\infty}^{-1} \begin{pmatrix} a_{rs} + z^{-j} p\, b_{rs} \\ 0 \end{pmatrix} z^s u^r \ .$$

But p contains only terms z^k for $k \leq j-1$, therefore $z^{-j}p$ contains only negative powers of z. Renaming the coefficients we may write

$$\sigma = \sum_{r=0}^{\infty} \sum_{s=-\infty}^{-1} \begin{pmatrix} a'_{rs} \\ 0 \end{pmatrix} z^s u^r$$

for some $a'_{rs} \in \mathbb{C}$, and consequently $T\sigma = \sum_{r=0}^{\infty} \sum_{s=-\infty}^{-1} \left(\begin{smallmatrix} z^j a'_{rs} \\ 0 \end{smallmatrix} \right) z^s u^r$. Here each term $a'_{rs} z^{j+s} u^r$ satisfying $j+s \leq kr$ is holomorphic on the V-chart. Subtracting these holomorphic terms we are left with an expression for a where the index s varies as $kr - j + 1 \leq s \leq -1$. This in turn forces $r \leq \left\lfloor \frac{j-2}{k} \right\rfloor$, giving the claimed expression for the 1-cocycle. □

Lemma 2.10. *Let E be the non-split bundle over Z_k represented in canonical form by (j,p) and $m > 0$ the smallest exponent of u appearing in p. If $\mu = \min\left(m, \left\lfloor \frac{j-2}{k} \right\rfloor\right)$, then*

$$l(R^1\pi_* E) \geq \mu \left(j - 1 - k\, \frac{\mu - 1}{2} \right) .$$

Proof. Assume $m \leq \left\lfloor \frac{j-2}{k} \right\rfloor$. Let $\sigma = (a,0)$ denote a 1-cocycle where $a = z^s u^r$ with $0 \leq r \leq m-1$ and $kr - j + 1 \leq s \leq -1$ (due to Lemma 2.9). We claim that σ represents a non-zero cohomology class. In fact, since σ is not holomorphic on U, for σ to be a coboundary there must exist a U-coboundary α such that $\sigma + \alpha$ is holomorphic on V. Hence, there must exist a polynomial X, holomorphic on U, making the expression $z^j a + pX$ holomorphic on V. However, $z^j a = z^{s+j} u^r$ is not holomorphic on V, since $s + j \geq kr + 1$. Moreover, by the choice of m, no term in pX cancels $z^j a$. Consequently, no choice of X solves the problem of holomorphicity on V. Hence $l(R^1\pi_* E)$ is at least the number of independent cocycles of the form $\sigma = (a,0)$, where $a = z^s u^r$ with $0 \leq r \leq m-1$ and $kr - j + 1 \leq s \leq -1$. But by exactly the same reasoning, any linear combination $p_{rs} z^s u^r$ (with r and s satisfying the above inequalities) is a coboundary if and only if every term is a coboundary individually, and hence the monomial cocycles $\sigma = (a,0)$ are linearly independent. There are $m\big((j-1) - k(m-1)/2\big)$ such terms.

On the other hand, if $m > \left\lfloor \frac{j-2}{k} \right\rfloor$, then by an analogous argument, none of the monomial terms in the canonical cocycle form of Lemma 2.9 are coboundaries, and so the same formula holds with m replaced by $\left\lfloor \frac{j-2}{k} \right\rfloor$. □

Lemma 2.11. *Let E satisfy the same conditions as in Lemma 2.10, and assume in addition that p is holomorphic on Z_k. Then*

$$l(R^1\pi_* E) = \mu \left(j - 1 - k\, \frac{\mu - 1}{2} \right) .$$

Proof. If p is holomorphic on Z_k, it can be written as

$$p(z,u) = \sum_{r=1}^{\lfloor (2j-2)/k \rfloor} \sum_{s=0}^{\min(j-1,\, kr)} p_{rs} z^s u^r .$$

Following the proof of Lemma 2.10, we now need to show that all remaining 1-cocycles, namely $\sigma = \big(a_{\bar r\bar s} z^{\bar s} u^{\bar r}, 0\big)$ with

$$m \le \bar r \le \left\lfloor \frac{j-2}{k} \right\rfloor \quad \text{and} \quad k\bar r - j + 1 \le \bar s \le -1 \tag{$*$}$$

are coboundaries. A fixed such 1-cocycle $\sigma = \big(a z^{\bar s} u^{\bar r}, 0\big)$ is a 1-coboundary if and only if there exist 0-cocycles $(\alpha, \beta) \in H^0(V; E)$ and $(x, y) \in H^0(U; E)$ such that

$$T^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \sigma + \begin{pmatrix}x\\ y\end{pmatrix} = 0 \ .$$

In other words, we need to find α, β holomorphic on V such that

$$A = \begin{pmatrix} z^{-j} & -p \\ 0 & z^j \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \begin{pmatrix} a z^{\bar s} u^{\bar r} \\ 0\end{pmatrix} = \begin{pmatrix} z^{-j}\alpha - p\beta + a z^{\bar s} u^{\bar r} \\ z^j \beta \end{pmatrix}$$

is holomorphic on U. Our goal is to cancel the coefficient a. Notice that, since $z^j a z^{\bar s} u^{\bar r}$ is not holomorphic on V, no choice of α can cancel a. So, we can assume $\alpha = 0$ and study the problem of holomorphicity on U for the simplified matrix

$$A = \begin{pmatrix} -p\beta + a z^{\bar s} u^{\bar r} \\ z^j \beta \end{pmatrix} \ .$$

Let $\sup\{l : p_{ml} \neq 0\} =: l_0 \ge 0$ by assumption, and set

$$\beta = \frac{a z^{\bar s} u^{\bar r}}{p_{ml_0} z^{l_0} u^m} = \frac{a}{p_{ml_0}} z^{\bar s - l_0} u^{\bar r - m} \ .$$

Since $\bar s - l_0 < 0$, β is holomorphic on V. Now the matrix A becomes

$$A = \begin{pmatrix} -\Big(\sum_{l=0}^{l_0-1} p_{ml} z^l u^m + p'\Big) \frac{a}{p_{ml_0}} z^{\bar s - l_0} u^{\bar r - m} \\ z^j \frac{a}{p_{ml_0}} z^{\bar s - l_0} u^{\bar r - m} \end{pmatrix} \ ,$$

where u^{m+1} divides p'. We have $l_0 \le \min(j-1,\ km)$ and $m \le \bar r$ by assumption. Using $(*)$, we obtain $j + \bar s - l_0 \ge 0$; consequently the second coordinate of A is holomorphic on U. The first coordinate of A is

$$-\sum_{l=0}^{l_0-1} \frac{a\, p_{ml}}{p_{ml_0}} z^{l+\bar s - l_0} u^{\bar r} - p' \frac{a}{p_{ml_0}} z^{\bar s - l_0} u^{\bar r - m} = -\sum_{l=-l_0}^{-1} p_{ml} z^{l+\bar s} u^{\bar r} - p' \frac{a}{p_{ml_0}} z^{\bar s - l_0} u^{\bar r - m} \ .$$

We obtain $A \sim -A^- - A^+$, where

$$A^- := \sum_{l=-l_0}^{-1} \begin{pmatrix} p_{ml} z^{l+\bar s} u^{\bar r} \\ 0 \end{pmatrix} \ , \quad A^+ := \begin{pmatrix} p' \frac{a}{p_{ml_0}} z^{\bar s - l_0} u^{\bar r - m} \\ 0 \end{pmatrix} \ .$$

This expresses σ as a linear combination of 1-cocycles having distinct degrees. Notice that A^- is either zero, or contains only powers of z strictly lower than $\bar{s}$; whereas A^+ is either zero or contains only powers of u strictly greater than $\bar{r}$.

To complete the proof we now proceed by finite induction on the exponents $\bar{r}$ and $\bar{s}$. The initial case is when $\bar{r} = m$ and $\bar{s} = km - j + 1$, and in this case it follows from Lemma 2.9 that $A^- \sim 0$, and we are left with A^+, whose degree in u is greater than m.

Now assume the lemma has been proved for cycles of the form $(a_{rs} z^s u^r, 0)$ for all $r < \bar{r}$ and $s < \bar{s}$ and apply the same algorithm to $\sigma = (a_{\bar{r}\bar{s}} z^{\bar{s}} u^{\bar{r}}, 0)$. Then again, A^- gets expressed as a combination of cocycles already known to be null-cohomologous, whereas A^+ gets expressed in terms of cycles of higher degree in u.

The induction finishes at $r = \lfloor (j-2)/k \rfloor$ when A^+ also becomes zero by Lemma 2.9. $\square$

Even though we did not compute explicitly the height for bundles whose extension class p is *not* holomorphic on Z_k (the computer algorithm can compute it for arbitrary p), we can show that the split bundle attains the maximally possible height for a fixed splitting type j:

Proposition 2.12. *Let $E(j,p)$ be the bundle of splitting type j whose extension class is given by p, and let $\bar{E}(j) := \mathcal{O}(-j) \oplus \mathcal{O}(j)$ denote the split bundle. If $u|p(z,u)$ and $p \not\equiv 0$, then*

$$\mathbf{h}_k\big(\bar{E}(j)\big) \geq \mathbf{h}_k\big(E(j,p)\big) \ .$$

Proof. Applying π_* to the short exact sequence

$$0 \longrightarrow \mathcal{O}(-j) \xrightarrow{\imath} E \longrightarrow \mathcal{O}(j) \longrightarrow 0 \tag{2.4}$$

gives the long exact sequence

$$\cdots \longrightarrow \pi_*\mathcal{O}(j) \xrightarrow{\delta} R^1\pi_*\mathcal{O}(-j) \xrightarrow{\imath_*} R^1\pi_* E \longrightarrow R^1\pi_*\mathcal{O}(j) \longrightarrow 0 \ ,$$

with $\pi_*\mathcal{O}(j) \neq 0$ because $j \geq 0$; and the connecting homomorphism δ is zero precisely when the sequence (2.4) splits. Hence, if E does not split,

$$\begin{aligned} l\big(R^1\pi_* E\big) &\leq l\big(\imath_* R^1\pi_*\mathcal{O}(-j)\big) + l\big(R^1\pi_*\mathcal{O}(j)\big) \\ &\leq l\big(R^1\pi_*\mathcal{O}(-j)\big) + l\big(R^1\pi_*\mathcal{O}(j)\big) \ , \end{aligned}$$

with equality holding for the split bundle (and possibly other bundles as well, for instance in cases when k does not divide j). $\square$

Remark 2.13. Let us stress again that it is necessary for u to divide p in order for the bundle $E(j,p)$ to split as $\mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$ on $\ell = \{u = 0\}$: For example, the bundle on Z_1 given by transition function $\left(\begin{smallmatrix} z^2 & z \\ 0 & z^{-2} \end{smallmatrix}\right)$, which we would denote by $E(2,z)$, is in fact not of splitting type 2, but 1:

$$\begin{pmatrix} 1 & 0 \\ -z^{-3} & 1 \end{pmatrix} \begin{pmatrix} z^2 & z \\ 0 & z^{-2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & z \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

2.2 Widths

Again we begin by computing the width of *line* bundles: Since computations get quite complicated very quickly, we chose to present one explicit example, and refer to the program for the general case.

Example 2.14 (Computation of $l(Q)$ for the bundle $\mathcal{O}(-3)$ over Z_2)**.**
Here we show that $\mathbf{w}_2\big(\mathcal{O}(-3)\big) = 0$:

In our canonical coordinates Z_2 is given by charts $U = \big\{(z,u)\big\}$, $V = \big\{(z^{-1}, z^2u)\big\}$, and the transition matrix for the bundle $E = \mathcal{O}(-3)$ is $T = \big(z^3\big)$. X_2 is the singular space obtained by the contraction of the zero section $\pi\colon Z_2 \to X_2$. The coordinate ring of X_2 is $\mathbb{C}[x,y,w]\big/(y^2 - xw)$, and the lifting map induced by π is (on the U-chart) $x \mapsto u$, $y \mapsto zu$ and $w \mapsto z^2u$.

Let $M = (\pi_* E)_0^\wedge$ denote the completion of the stalk $(\pi_* E)_0$. Let $\rho\colon M \hookrightarrow M^{\vee\vee}$ denote the natural inclusion of M into its double dual. We want to compute $l(Q) = \dim(\operatorname{coker}\rho)$. By the Theorem on Formal Functions,

$$M \cong \varprojlim H^0\big(\ell_n, E|_{\ell_n}\big) \ ,$$

where ℓ_n denotes the n^{th} infinitesimal neighbourhood of ℓ. To determine M, it suffices by Lemma 2.5 to calculate $H^0\big(\ell_n, E|_{\ell_n}\big)$ for a fixed $n \geq N = \lfloor (2j-2)/k \rfloor$ (cf. also [BG2] or [Ga1]) and the relations among its generators under the action of $\mathcal{O}_0^\wedge \cong \mathbb{C}[[x,y,w]]\big/(y^2 - xw)$. In this example, a section σ of E on the U-chart has the expression

$$\sigma = \sum_{r=0}^{N} \sum_{s=0}^{\infty} a_{rs} z^r u^s \ ,$$

and changing coordinates, we have the condition that

$$T\sigma = z^3 \sigma$$

must be holomorphic on $V = \big\{(z^{-1}, z^2u)\big\}$, and hence must have only terms of the form $z^s u^r$ that satisfy $2s \leq r$, with the remaining coefficients vanishing. It follows that the expression for σ has the form

$$\begin{aligned} \sigma &= \sum_{r=2}^{N} \sum_{s=0}^{\left\lfloor \frac{2r-3}{2} \right\rfloor} a_{rs} z^s u^r \\ &= a_{20}u^2 + a_{21}zu^2 + a_{30}u^3 + a_{31}zu^3 + a_{32}z^2u^3 + a_{33}z^3u^3 + \cdots \ . \end{aligned}$$

Consequently, all terms of σ are generated over $\mathcal{O}_0^\wedge$ by $\beta_0 = u^2 = x^2$ and $\beta_1 = zu^2 = xy$. We obtain the $\mathcal{O}_0^\wedge$-module $M \cong \mathcal{O}_0^\wedge\big[\beta_0, \beta_1\big]\big/\mathbf{R^1}$, where $\mathbf{R^1}$ is the set of relations

$$\mathbf{R^1} = \begin{cases} R_1^1 : & \beta_0 y - \beta_1 x \ , \\ R_2^1 : & \beta_0 w - \beta_1 y \ . \end{cases}$$

Consequently, the dual is $M^\vee = \mathcal{O}_0^\wedge\big[\beta_0^\vee, \beta_1^\vee\big]/\mathbf{S}$, where

$$\beta_0^\vee = \begin{cases} \beta_0 \mapsto x \\ \beta_1 \mapsto y \end{cases} , \qquad \beta_1^\vee = \begin{cases} \beta_0 \mapsto y \\ \beta_1 \mapsto w \end{cases} ,$$

and $\mathbf{S}$ is the set of relations

$$\mathbf{S} = \begin{cases} S_1 : & \beta_0^\vee y - \beta_1^\vee x \ , \\ S_2 : & \beta_0^\vee w - \beta_1^\vee y \ . \end{cases}$$

Clearly, $M \cong M^\vee$, consequently $\rho \colon M \hookrightarrow M^{\vee\vee}$ is also an isomorphism, so $\mathbf{w}_2\big(\mathcal{O}(-3)\big) = 0$. //

We remark as an aside that there is another set of relations $\mathbf{R^2}$ among the relations of M:

$$\mathbf{R^2} = \begin{cases} R_1^2 : & R_1^1 y - R_2^1 x \ , \\ R_2^2 : & R_1^1 w - R_2^1 y \ ; \end{cases}$$

and so on:

$$\mathbf{R^n} = \begin{cases} R_1^n : & R_1^{n-1} y - R_2^{n-1} x \ , \\ R_2^n : & R_1^{n-1} w - R_2^{n-1} y \ . \end{cases}$$

This is an example of a theorem of Eisenbud [Ei], which says that every minimal resolution of a finitely generated module over $A := R/(x)$ becomes periodic with period 1 or 2 after at most $\dim A$ steps, where R is a regular ring and x a non-unit.

Lemma 2.15. *Assume $j \geq 0$ and let $n_2 = \left\lfloor \frac{j}{k} \right\rfloor$. Then*

$$\begin{aligned} \mathbf{w}_k\big(\mathcal{O}(j)\big) &= (j+1)n_2 - kn_2(n_2+1)/2 \ , \\ \mathbf{w}_k\big(\mathcal{O}(-j)\big) &= 0 \ . \end{aligned}$$

Proof. The length $l(Q)$ equals the dimension of $Q_0^\wedge$ as a $\mathbb{C}$-vector space, where $0 \in X_k$ is the singular point. Since Q is defined by the sequence (1.1), $Q_0^\wedge$ is the cokernel of the map $\big(\pi_*\mathcal{O}(j)\big)_0^\wedge \to \big(\pi_*\mathcal{O}(j)\big)_0^{\vee\vee\wedge}$. By the Theorem on Formal Functions [Ha, p. 276],

$$\big(\pi_*\mathcal{O}(j)\big)_0^\wedge = \varprojlim H^0\Big(\ell_n; \mathcal{O}(j)\big|_{\ell_n}\Big) =: M \ .$$

But the limit stabilises at a finite stage by Lemma 2.5, so it suffices to compute the group $H^0\big(\ell_n; \mathcal{O}(j)|_{\ell_n}\big)$ for large n. We first compute the k_0-module structure on $M := (\pi_*\mathcal{O}(j))_0^\wedge$. Note that here

$$k_0 \cong \mathbb{C}[[x_0, x_1, \ldots, x_k]]/\big\{x_i x_j - x_{i+1}x_{j-1}\big\} \ \text{ for } i = 0, 1, \ldots, k-2 \ , \ j = i+2, \ldots, k \ ,$$

and the contraction map $\pi \colon Z_k \to X_k$ is given in (z,u)-coordinates by $x_i = z^i u$.

In the case of $\mathcal{O}(j)$, M is generated as a k_0-module by the monomials $\beta_i = z^i$ for $0 \leq i \leq j$ with relations $\beta_i x_{s-1} - \beta_{l-1} x_l = 0$ for $1 \leq i \leq j$ and $1 \leq l \leq k$. Now, one can use any computer algebra package (e.g. *Macaulay 2* [M2], *Maple*, *Magma*, see [GS]) to compute

$M^{\vee\vee}/M$: Writing $j = mk + b$ with $1 \le b \le k$ (so that $m = \lfloor (j-1)/k \rfloor$), one finds that $M^\vee$ and $M^{\vee\vee}$ are generated by $k - b + 1$ elements and that the cokernel has, as a $\mathbb{C}$-vector space, a basis consisting of all the monomials of degree $\le m$. Accounting for the relations coming from $M^{\vee\vee}$ and from the ground ring k_0 shows that the dimension of the cokernel is $(j+1)n_2 - kn_2(n_2+1)/2$.

In the case $\mathcal{O}(-j)$, set $\nu = -j \mod k$, so that $-j = -qk + \nu$ with $0 \le \nu < k$. Then for large n, $H^0\bigl(\ell_n; \mathcal{O}(j)|_{\ell_n}\bigr)$, and hence M, is generated by the set of monomials $\alpha_i = z^i u^q$, for $0 \le i \le \nu$, with relations $\alpha_i x_{l-1} - \alpha_{i-1} x_l = 0$ for $1 \le i \le \nu$ and $1 \le l \le k$. By direct calculation, or using a computer algebra package, one shows that the evaluation map $M \to M^{\vee\vee}$ is an isomorphism, so $\mathbf{w}_k(\mathcal{O}(-j)) = 0$. Explicit computation for $\mathcal{O}(-2)$ is given in Example 2.14. $\square$

Theorem 2.16. *Let E be a rank-2 bundle over Z_k of splitting type j. Then the following bounds are sharp: For $j > 0$ and with $n_2 = \lfloor \frac{j}{k} \rfloor$,*

$$0 \le \mathbf{w}_k(E) \le (j+1)n_2 - kn_2(n_2+1)/2 \text{ , and } \mathbf{w}_1(E) \ge 1 \text{ .}$$

Furthermore, for all $0 < j < k$, $\mathbf{w}_k(E) = 0$ for all bundles E (and necessarily $k > 1$).

Proof. The upper bound is realised by the split bundle, so the right-hand side follows from Lemma 2.15 and Proposition 2.3. This follows from direct computation: First we write down a set of possible generators of M, then we compute their relations. But relations can only come from the off-diagonal part of the transition function of E, i.e. from p, so the split bundle with $p = 0$ has the fewest possible relations in M, hence the biggest cokernel $M^{\vee\vee}/M$.

To calculate the lower bound for $\mathbf{w}_k(E)$, note that by definition $\mathbf{w}_k(E) \ge 0$. But for $k > 1$, the bundle given in canonical coordinates by transition matrix $\left(\begin{smallmatrix} z^j & zu \\ 0 & z^{-j} \end{smallmatrix}\right)$ has width zero. Direct computations as described in [Ga3] show that for the case $k = 1$, $\mathbf{w}_1(E) = 1$. $\square$

Remark 2.17. It is interesting to notice that on the space Z_1 any nontrivial rank-2 bundle E with $c_1(E) = 0$ has $\mathbf{w}_1(E) \ne 0$. This is in strong contrast with what happens on Z_k for $k > 1$.

2.3 Local holomorphic Euler characteristic

By summing up the results for heights and widths and using Definition 1.3 we obtain bounds on the local Euler characteristic of a rank-2 bundle E on Z_k near ℓ. Due to the occurrence of integer part functions, it will be useful express the splitting type j of E as $j = nk + b$, with $0 \le b < k$.

Corollary 2.18. *Let E be a rank-2 bundle over Z_k of splitting type j with $j > 0$ and let $j = nk + b$ as above. The following are sharp bounds for the local holomorphic Euler characteristic of E:*

$$j - 1 \le \chi(\ell, E) \le \begin{cases} n^2k + 2nb + b - 1 & \text{if } k \ge 2 \text{ and } 1 \le b < k \text{ ,} \\ n^2k & \text{if } k \ge 2 \text{ and } b = 0 \text{ ,} \end{cases}$$

and

$$j \le \chi(\ell, E) \le j^2 \text{ for } k = 1 \text{ .}$$

Proof. It follows directly from the definitions that for a curve $\ell^2 = -k$ inside a smooth surface, $h^0\big(X; (\pi_* E)^{\vee\vee}/\sigma_* E\big) = \mathbf{w}_k(E)$ and $h^0(X; R^1\sigma_* E) = \mathbf{h}_k(E)$, so the result follows by adding the results of Theorems 2.8 and 2.16.

To aid the computation, note that for $2 \leq b < k$ we have $\lfloor \frac{j}{k} \rfloor = \lfloor \frac{j-2}{k} \rfloor = n$, while for $b = 0, 1$ we have $\lfloor \frac{j}{k} \rfloor = n$ and $\lfloor \frac{j-2}{k} \rfloor = n - 1$. We find that for $b = 0, 1$, $\chi(\ell, E) \leq n^2 k + 2nb$, and we can absorb the case $b = 1$ into the other case.

The upper bound is attained by the split bundle $E = \mathcal{O}(-j) \oplus \mathcal{O}(j)$, while the lower bound is attained by the "generic" bundle with extension $p(z, u) = zu$ as in the proof of Lemma 2.15. $\square$

We intend to study numerical invariants of bundles over more general exceptional loci in future papers. One property of local holomorphic Euler characteristics of *instantons* (where $j = nk$ for some n) is immediate from Corollary 2.18:

Corollary 2.19. *For a rank-2 bundle E on Z_k with $c_1(E) = 0$ and splitting type $j = nk$, the integer ranges $[1, \dots, k-2]$ and $[k+1, \dots, 2k-2]$ cannot occur as local holomorphic Euler characteristics ("instanton charges"). These ranges are non-empty when $k > 2$. (E.g. on Z_3, charges 1 and 4 cannot occur.*[2]*)*

3 Balancing

We now consider the question of constructing vector bundles with specified numerical invariants. We use the technique of *balancing bundles*. This technique was used in [BG2] to prove the existence of bundles over Z_1 with any prescribed numerically admissible invariants, and in [BG3] to study bundles on Z_2.

Given two bundles E and E' of splitting type $(j_1, \dots, j_r)$ and $(j'_1, \dots, j'_r)$, we say that E is *more balanced* than E' if $j_1 - j_r \leq j'_1 - j'_r$. The advantage of balancing a bundle is that we control the numerical invariants at each step, and we only need to compute numerical invariants for a smaller range of bundles.

The simplest case of balancing is for rank-2 bundles and goes as follows. If $j_1 - j_2 \leq k - 1$, we have won and we stop. If $j_2 \leq j_1 - k$ we make the construction of [BG3], namely an elementary transformation with respect to $\mathcal{O}_\ell(j_2)$ and obtain a new more balanced bundle with splitting type $(j_1, j_2 + k)$. We may also compare the invariants of the two bundles; at the end we reduce to a case with $j_1 - j_2 \leq k - 1$. We now describe how to balance bundles of rank $r \geq 2$. Let E be a rank-r vector bundle over Z_k of splitting type $j_1 \geq j_2 \geq \cdots \geq j_r$. We say that E is *balanced* if $j_1 \leq j_r + k - 1$. The objective is to balance E. Balancing associates to E the following data:

1. A positive integer t (the number of steps);

2. a finite sequence of r-tuples of non-increasing integers $\{j(i, l)\}$ with $1 \leq i \leq t$ and $1 \leq l \leq r$ (the splitting types) satisfying:

[2] But the bundle with $p(z, u) = z^{-1}u + z^4 u^2$ has charge 7. Also, it seems always possible to create a new bundle with charge $+k$ via elementary transformations, and since charges 2 and 3 exist by Corollary 2.18, we expect these are the only charge gaps on Z_3.

(a) $j(1,l) = j_l$ for $1 \le l \le r$ (the splitting of the bundle E),

(b) $\sum_{l=1}^{r} j(i,l) = \sum_{i=1}^{r} j(1,l) + ki - k$ for $2 \le i \le t$ (change of splitting produced by an elementary transformation), and

(c) $j(t,1) \le j(t,r) + k - 1$ (arrive at a balanced bundle);

3. a chain $E_1, \ldots, E_t$ of vector bundles, starting with $E_1 = E$, where E_i has splitting type $j(i,1) \ge j(i,2) \ge \cdots \ge j(i,r)$ for $1 \le i \le t$.

Definition 3.1. A sequence $\{j(i,l)\}$ of r-tuples of integers satisfying the numerical properties 1, 2 and 3 above is called an *admissible sequence*. The sequence $\{j(i,l)\}_{i=1}^{t}$ of splitting types of the bundles E_i obtained in balancing the bundle E is then called the admissible sequence *associated* to E.

Balancing a bundle E proceeds as follows: Set $E_1 := E$ and $j(1,l) = j_l$ for $1 \le l \le r$. If $j(1,1) \le j(1,r) + k - 1$ we have won, we set $t := 1$ and stop. Otherwise, $j(1,1) \ge j(1,r) + k$. Choose a surjective homomorphism $\rho\colon E|_\ell \to \mathcal{O}_\ell(j_r)$ and make the corresponding elementary transformation $\mathbf{r}\colon E \to \mathcal{O}_\ell(j_r)$. Set $E_2 := \ker \mathbf{r}$. Since $j_r \le j_{r-1}$, we have $\ker \rho \cong \bigoplus_{i=1}^{r-1} \mathcal{O}_\ell(j_i)$, and $\ker \mathbf{r}|_\ell$ fits in the exact sequence

$$0 \to \mathcal{O}_\ell(j_r + k) \to \ker \mathbf{r}|_\ell \to \bigoplus_{i=1}^{r-1} \mathcal{O}_\ell(j_i) \to 0 \ . \tag{3.1}$$

We call $j(2,1) \ge j(2,2) \ge \cdots \ge j(2,r)$ the *splitting type* of E_2. In particular,

$$\sum_{l=1}^{r} j(2,l) = \sum_{l=1}^{r} j(1,l) + k \ .$$

If $j(2,1) \le j(2,r) + k - 1$, then we have won and we stop. If $j(2,1) \ge j(2,r) + k$, then we obtain in the same way a bundle E_3 with splitting type $j(3,1) \ge \cdots \ge j(3,r)$ such that $j(3,1) \le j(2,1) \le j(1,1)$, $j(3,r) \ge j(2,r) \ge j(1,r)$. We continue this process, thus obtaining bundles $E_2, E_3, \ldots E_i, \ldots$ with splitting type $j(i,1) \ge \cdots \ge j(i,r)$, $j(i,1) \le j(i-1,1)$, $j(i,r) \ge j(i-1,r)$ and $\sum_{l=1}^{r} j(i,l) = \sum_{l=1}^{r} j(1,l) + ki - k$ for all integers $i \ge 2$ for which the bundle E_i is defined. We must show that this procedure stops, i.e. that after finitely many steps we arrive at a bundle E_i such that $j(i,1) \le j(i,r) + k - 1$. At the same time we will show that, in a suitable sense, each bundle E_x, $x \ge 2$, which is defined is more balanced than the preceding one E_{x-1}. Call $a_1 \ge \cdots \ge a_r$ the splitting type of the bundle $T := \mathcal{O}_\ell(j_r + k) \oplus \bigoplus_{i=1}^{r-1} \mathcal{O}_\ell(j_i)$. Since $j_1 \ge j_r + k$ by assumption, we have $a_1 = j_1$. If $j_{r-1} \ge j_r + k$, then $a_r = j_r + k$ and hence $a_1 - a_r < j_1 - j_r$. If $j_{r-1} \le j_r + k - 1$, then $a_r = j_{r-1}$ and hence $a_1 - a_r = j_1 - j_{r-1}$. Hence in the latter case we have $a_1 - a_r < j_1 - j_r$, unless $j_{r-1} = j_r$. By the exact sequence (3.1) and semi-continuity, the bundle $E_2|_\ell$ is more balanced (in the sense of the Harder–Narasimhan filtration) than the bundle T. Hence we always have $j(2,1) - j(2,r) \le j(1,1) - j(1,r)$, while $j(2,1) - j(2,r) < j(1,1) - j(1,r)$ unless $j(1,r-1) = j(1,r)$. Even in this case, since $\sum_{l=1}^{r} j(x,l) = k + \sum_{l=1}^{r} j(x-1,l)$, we see that the procedure must stop after finitely many steps.

We now construct families of bundles having prescribed associated admissible sequence, generalising [BG3, Theorem 2.1], which constructed such families for rank-2 bundles near a -1-curve. The generalisation requires only minor modifications, but we give the details for completeness.

Theorem 3.2. *Fix an admissible sequence $\{j(i,l)\}_{i=1}^{t}$ and let E and F be rank-r vector bundles on $\widehat{\ell}$ with $\{j(i,l)\}_{i=1}^{t}$ as an associated admissible sequence. Then there exists a flat family $\{E_s\}_{s\in T}$ of rank-r vector bundles on $\widehat{\ell}$ parametrised by an integral variety T and $s_0, s_1 \in T$ with $E_{s_0} \cong E$ and $E_{s_1} \cong F$ such that E_s has $\{j(i,l)\}_{i=1}^{t}$ as admissible sequence for every $s \in T$.*

Proof. We use induction on t. If $t = 1$ the result is obvious because E and F are split vector bundles with the same splitting type and are hence isomorphic. Assume that $t > 1$ and that the result is true for $t-1$. Let E_2, F_2 be the second bundles associated to E, F respectively. Hence E_2 and F_2 have $\{j(2,l)\}_{i=2}^{t}$ as an associated admissible sequence. By induction, there is a flat family $\{E'_s\}_{s\in S}$ of rank-r vector bundles on $\widehat{\ell}$ and $m_0, m_1 \in S$ with $E'_{m_0} \cong E_2$, $E'_{m_1} \cong F_2$ and such that E'_s has $\{j(i,l)\}_{i=2}^{t}$ as an associated admissible sequence for every $s \in S$. We write $j'_i = j(2,i)$ to simplify notation. By the balancing construction, the bundles E and F fit into exact sequences

$$0 \to E \to E_2(\ell) \to \mathcal{O}_\ell(j_1 - k) \to 0$$
$$0 \to F \to F_2(\ell) \to \mathcal{O}_\ell(j_1 - k) \to 0$$

For every bundle M on $\widehat{\ell}$ having $\{j(i,l)\}_{i=2}^{t}$ as an associated admissible sequence, the set of surjective homomorphisms $\mathbf{t}\colon M(\ell) \to \mathcal{O}_\ell(j_1 - k)$ is parametrised by an integral variety whose dimension depends only on j_1, j_2 and $j'_2 = j_1 + j_2 - j'_1 + k$. The kernel $\ker \mathbf{t}|_\ell$ is an extension of $\mathcal{O}_\ell(j'_1)$ by $\mathcal{O}_\ell(j_1)$. This extension splits since $j_1 \geq j'_1 + k$, and hence the bundle $\ker \mathbf{t}$ has $\{j(i,l)\}_{i=1}^{t}$ as an admissible sequence. Varying M among bundles E'_s for $s \in S$ we get that the set of all such surjections is parametrised by an irreducible non-empty variety T. For any fixed ample line bundle H on the n^{th} neighbourhood of ℓ, it follows from the exact sequences in the balancing construction that the bundles in this family have the same Hilbert polynomial with respect to H, and therefore the family is flat. □

4 Moduli

Definition 4.1. We write

$$\mathcal{M}_j(k) = \big\{E \to Z_k : E|_\ell \cong \mathcal{O}(j) \oplus \mathcal{O}(-j)\big\}/\!\sim$$

for the moduli stack of bundles over Z_k of splitting type j.

Recall from (2.2) that a bundle E on Z_k of splitting type j is an extension of $\mathcal{O}(j)$ by $\mathcal{O}(-j)$ and is therefore determined by its extension class. In our choice of coordinates, this amounts to saying that E is determined by the pair (j,p), where j is the splitting type and p is a polynomial as in (2.3). Let $N := \lfloor \frac{2j-2}{k} \rfloor$ as before; then p has $m := N(2j-1) - k\binom{N+1}{2}$

coefficients. We identify p as an element in $\mathbb{C}^m$ by writing its coefficients in lexicographical order. We then define the equivalence relation $p \sim p'$ if (j,p) and (j,p') define isomorphic bundles over Z_k. We give $\mathbb{C}^m$ the quotient topology. There is a bijection

$$\begin{array}{rcl} \phi\colon \mathcal{M}_j(k) & \to & \mathbb{C}^m/\!\sim \ , \\ \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} & \mapsto & \bigl(p_{1,k-j+1},\ldots,p_{N,j-1}\bigr) \ , \end{array}$$

where

$$p(z,u) = \sum_{r=1}^{N} \sum_{s=kr-j+1}^{j-1} p_{rs}\, z^s u^r \ .$$

We give $\mathcal{M}_j(k)$ the topology induced by this bijection. Here are some examples:

Example 4.2. For each k, $\mathcal{M}_0(k)$ contains only one point, corresponding to the trivial bundle over Z_k. In other words, if a bundle over Z_k is trivial over the zero section, it is globally trivial.

Example 4.3. For each $2j-2<k$, $\mathcal{M}_j(k)$ contains only one point. In other words, a holomorphic bundle over Z_k of splitting type $2j-2<k$ splits. This can be verified directly from formula (2.3).

Example 4.4. $\mathcal{M}_2(2)$ contains exactly two points, [So, Theorem 6.24].

Example 4.5. $\mathcal{M}_2(1) \simeq \mathbb{P}^1 \cup \{A,B\}$, where A and B are points, with open sets $U \subset \mathbb{P}^1$, where U is open in the usual topology of $\mathbb{P}^1$, $\mathbb{P}^1 \cup \{A\}$, and the whole space, [Ga2, Theorem 4.2].

Example 4.6. $\mathcal{M}_3(2) \simeq \mathbb{P}^2 \cup \{A,B\}$, where A and B are points, with open sets $U \subset \mathbb{P}^2$, where U is open in the usual topology of $\mathbb{P}^2$, $\mathbb{P}^2 \cup \{A\}$, and the whole space, [So, Theorem 6.35].

Example 4.7. $\mathcal{M}_j(k)$ is non-Hausdorff for $2j-2 \geq k$. This uses Theorem 4.11 below to show non-emptiness (positive dimensionality if the inequality is strict). Notice that the only open neighbourhood of the split bundle is the entire $\mathcal{M}_j(k)$.

Lemma 4.8. *If $p' = \lambda p$ for some $\lambda \in \mathbb{C}^\times$, then the matrices*

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix}$$

give holomorphically equivalent vector bundles.

Proof. Just write down the isomorphism as

$$\begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda \end{pmatrix} \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \ .$$

□

Notation: We write $\mathcal{F}^{(n)} := \mathcal{F}|_{\ell_n} = \mathcal{F} \otimes \mathcal{O}/\mathcal{I}_\ell^{n+1}$ for any coherent sheaf $\mathcal{F}$ and similarly $E^{(n)} := E|_{\ell_n}$ for vector bundles on Z_k.

Theorem 4.9. *On the first infinitesimal neighbourhood, two bundles $E^{(1)}$ and $E'^{(1)}$ with respective transition matrices*

$$\begin{pmatrix} z^j & p_1 \\ 0 & z^{-j} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} z^j & p_1' \\ 0 & z^{-j} \end{pmatrix}$$

are isomorphic if and only if $p_1' = \lambda p_1$ for some $\lambda \in \mathbb{C}^\times$.

Proof. The "if" part is Lemma 4.8. Now suppose $E^{(1)}$ and $E'^{(1)}$ are isomorphic. According to our notation we have $p_1 = \sum_{s=k-j+1}^{j-1} p_{1s}\, z^s\, u$ and $p_1' = \sum_{s=k-j+1}^{j-1} p_{1s}'\, z^l\, u$. We will write the isomorphism in the form

$$\begin{pmatrix} z^j & p_1' \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z^j & p_1 \\ 0 & z^{-j} \end{pmatrix} ,$$

where a, b, c and d are holomorphic on U, and α, β, γ and δ are holomorphic on V. On the first infinitesimal neighbourhood, this yields the following set of equations:

$$\begin{cases} \big(a_0(z) + a_1(z)u\big)z^j + p_1' c_0(z) = \big(\alpha_0(z^{-1}) + \alpha_1(z^{-1})zu\big)z^j & \text{(A1)} \\ z^{-j}\big(c_0(z) + c_1(z)u\big) = \big(\gamma_0(z^{-1}) + \gamma_1(z^{-1})zu\big)z^j & \text{(A2)} \\ \big(b_0(z) + b_1(z)u\big)z^j + p_1' d_0(z) = \alpha_0(z^{-1})p_1 + \big(\beta_0(z^{-1}) + \beta_1(z^{-1})zu\big)z^{-j} & \text{(A3)} \\ z^{-j}\big(d_0(z) + d_1(z)u\big) = \gamma_0(z^{-1})p_1 + \big(\delta_0(z^{-1}) + \delta_1(z^{-1})zu\big)z^{-j} & \text{(A4)} \end{cases}$$

Recalling that p_1 and p_1' are multiples of u and equating terms that are independent of u in (A1) and (A4) gives $a_0(z) = \alpha_0(z^{-1})$ and $d_0(z) = \delta_0(z^{-1})$ respectively. Therefore a_0, α_0, d_0 and δ_0 are constants, and $a_0 = \alpha_0$ and $d_0 = \delta_0$. Next we equate terms in u in Equation (A3), obtaining

$$b_1(z)u\, z^j + p_1' d_0 = \alpha_0 p_1 + \beta_1(z^{-1})u\, z^{-j} .$$

In Equation (A3), $z^j\, b_1$ has only terms z^l for $l \geq j$, and $z^{-j}\, \beta_1$ has only terms z^l for $l \leq -j$. Consequently, they do not affect the terms appearing in p_1 and p_1'; and the remaining part of Equation (A3) gives $p_1' d_0 = \alpha_0 p_1$. We observe that p_1 and p_1' differ by a constant.

It remains to show that d_0 and α_0 are non-zero. Taking terms that are independent of u in Equation (A3) we have $b_0(z)\, z^j = \beta_0(z^{-j})\, z^{-j}$, which implies $b_0(z) = \beta_0(z^{-1}) = 0$. It follows that over the exceptional divisor our coordinate change has determinant $a_0\, d_0$, hence $\alpha_0\, \delta_0 = a_0\, d_0 \neq 0$. □

Remark 4.10 (Construction of the generic set and further strata)**.** For a fixed bundle E over Z_k with transition matrix $\left(\begin{smallmatrix} z^j & p \\ 0 & z^{-j} \end{smallmatrix}\right)$, we write any automorphism of E in the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z^{-j} & -p \\ 0 & z^j \end{pmatrix} , \tag{4.1}$$

where a, b, c, d are holomorphic on U and α, β, γ, δ are holomorphic on V. This gives rise to a set of four equations, imposing conditions on a, b, c, d that make the expression on the right-hand side holomorphic on V (see the Long Proof of Theorem 4.11). Therefore, the set of automorphisms of E is determined by the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ subjected to certain constraints for holomorphicity. The number of constraints depends on p, and for a fixed j there is a minimal number of constraints that happens for "most" of the bundles. We call *generic* a bundle whose automorphisms realise the minimal number of constraints. It would be desirable to phrase this as "a bundle is generic (or, even better, stable) when it has the smallest number of automorphisms" as in the compact case; alas, the spaces of automorphism for each bundle is infinite-dimensional; moreover, no concept of stability is available for bundles on non-compact spaces.

Take for example the extreme cases: for a fixed j we have $p(z,u) = \sum p_{rs}\, z^s u^r$, and the two opposite cases are (i) when $p_{rs} \neq 0\ \forall r, s$ and the corresponding bundle is generic; and (ii) when $p = 0$ and the corresponding bundle belongs to the least generic stratum. What is happening is the simple fact that extra conditions on p make the equations on a, b, c, d harder to solve.

One can see the division of $\mathcal{M}_j(k)$ into strata as a constructive algorithm that proceeds by infinitesimal neighbourhoods. Start with the first infinitesimal neighbourhood, where we know that the only automorphisms are scalar multiplication. Now, pass to the second infinitesimal neighbourhood and keep in the generic stratum all bundles having the most automorphisms, that is, the ones whose expression of p is such that the minimal number of constraints is imposed on a, b, c, d. Separate the other bundles away form the generic stratum, again divided by number of automorphisms. Now restart the process on the third neighbourhood, and so on. When this process is carried out on the neighbourhood $N = \lfloor (2j-2)/k \rfloor$, then we are done, so it is a finite procedure. This process fixes the bundle $\operatorname{End}(E)$ and therefore its cohomologies $H^i\big(\mathscr{E}nd(E)\big)$. In particular $H^1\big(\mathscr{E}nd(E)\big)$ is fixed, and so is $H^1(E)$ and correspondingly the height $\mathbf{h}_k(E)$. Therefore only bundles with the same height belong to the same stratum (but not conversely). The generic stratum is contained in the set of bundles with minimal $h^1\big(\mathscr{E}nd(E)\big)$ (like stable points in the compact case), and hence minimal height. Now, $H^0\big(\mathscr{E}nd(E)\big)$ is also fixed by the N^{th} neighbourhood, and so is $H^0(E)$. This is not quite the same as fixing the width, though. We have

$$0 \to S^n(N^*_{\ell,Z_k}) \to \mathcal{O}^{(n)} \to \mathcal{O}^{(n-1)} \to 0 \ ,$$

where we write N_{ℓ,Z_k} for the normal sheaf of $\ell \subset Z_k$, and tensoring with E gives

$$0 \to S^n(N^*_{\ell,Z_k}) \otimes E \to E^{(n)} \to E^{(n-1)} \to 0 \ .$$

Now taking the direct image under π gives

$$0 \to \pi_*\big(S^n(N^*_{\ell,Z_k}) \otimes E\big) \to \pi_*\big(E^{(n)}\big) \to \pi_*\big(E^{(n-1)}\big) \to R^1\pi_*\big(S^n(N^*_{\ell,Z_k}) \otimes E\big) \to \cdots \ .$$

Now for sufficiently high n, $\pi_* E^{(n)}$ completely determines the width. We can compare the corresponding matrices of endomorphisms for two bundles E and E'. Suppose they have the same width, but yet $H^0\big(\mathscr{E}nd(E)^{(n)}\big)$ and $H^0\big(\mathscr{E}nd(E')^{(n)}\big)$ have positive relative dimension. Here, because the canonical expressions of extension classes are cut off at u-degree $N = \lfloor (2j-2)/k \rfloor$, we cannot observe differences happening at intervals smaller than k, so we must now restrict ourselves to the instanton case, where $j = nk$, and correspondingly we can observe the relative values of h^0 properly. Comparing the above sequence with $\mathscr{E}nd(E)$ and $\mathscr{E}nd(E')$ in place of E, we see that the only way this can happen is that also the $R^1\pi_*$-terms have different relative dimensions, but in this case the heights are different, and the bundles belong to different strata.

Theorem 4.11. *For $j \geq k$, $\mathcal{M}_j(k)$ has an open, dense subspace homeomorphic to a complex projective space $\mathbb{P}^{2j-2-k}$ minus a closed subvariety of codimension at least $k+1$.*

Short proof. [Ga2, Theorem 3.5] showed that the generic set of $\mathcal{M}_j(1)$ is a projective space $\mathbb{P}^{2j-3}$ minus a closed subvariety of codimension ≥ 2. The only modification needed to generalise the proof to $k > 1$ is the calculation of dimension of the generic set. Generic bundles do not split on the first infinitesimal neighbourhood, and there the only equivalence relation is projectivisation, by Theorem 4.9. The dimension count follows from formula (2.3), which shows that the u-coefficients are $\sum_{s=k-j+1}^{j-1} p_{1s}$. There are $2j-k-1$ coefficients, and after projectivising we obtain $\mathbb{P}^{2j-k-2}$. However, not all points of this $\mathbb{P}^{2j-k-2}$ are generic, and one must remove a closed subvariety where too much vanishing of coefficients occurs, thus implying larger numerical invariants. The closed subvariety to be removed contains all points having coefficients $z^s u = 0$ for $0 \leq s \leq k$, hence is defined by at least $k+1$ equations. $\square$

Long proof. Let E and E' be the bundles on Z_k given respectively by the extension classes p and p'. By Theorem 4.9 we know that on ℓ_1 the only isomorphism of bundles is scaling, so we may assume

$$p = p_1 + p_2 \qquad \text{and} \qquad p' = p_1 + p_2' \ ,$$

where $p_1 := p|_{\ell_1}$. If E and E' are isomorphic, then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z^{-j} & -p \\ 0 & z^j \end{pmatrix} = \begin{pmatrix} a + z^{-j}p'c & z^{2j}b + z^j(dp' - ap) - cpp' \\ z^{-2j}c & d - z^{-j}pc \end{pmatrix} , \tag{4.2}$$

for some change of coordinates $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ holomorphic on U and $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ holomorphic on V, which we may assume have determinant one. We write

$$\alpha(z^{-1}, z^k u) = \alpha_0(z^{-1}) + \alpha_1(z^{-1})\, z^k u + \cdots \text{ , similarly for } \beta,\ \gamma,\ \delta, \text{ and}$$

$$a(z,u) = a_0(z) + a_1(z)u + a_2(z)u^2 + \cdots \text{ , similarly for } b,\ c,\ d,$$

where the coefficients are convergent power series in z^{-1} or z, respectively. Then

$$\begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 z^{2j} \\ c_0 z^{-2j} & d_0 \end{pmatrix} ,$$

implies $\beta_0 = b_0 = 0$, and $c_0(z) = c_{00} + c_{01}z + \cdots + c_{0,2j}z^{2j}$. It follows that $\alpha_{00} = a_{00} = \lambda = \delta_{00}^{-1} = d_{00}^{-1} = 1$.

The coefficients of u are

$$\begin{pmatrix} \alpha_1 z^k u & \beta_1 z^k u \\ \gamma_1 z^k u & \delta_1 z^k u \end{pmatrix} = \begin{pmatrix} a_1 u + p_1 c_0 z^{-j} & b_1 u z^{2j} + z^j\cancel{(d_0 p_1 - a_0 p_1)} \\ c_1 u z^{-2j} & d_1 u - p_1 c_0 z^{-j} \end{pmatrix} , \tag{4.3}$$

which has to be holomorphic in $(z^{-1}, z^k u)$. This forces $b_1(z) = b_{10}+b_{11}z+\cdots+b_{1,-2j+k}z^{-2j+k}$, whence $b_1 \neq 0$ provided $k-2j \geq 0$, i.e. $j \leq \lfloor \frac{k}{2} \rfloor$. But by assumption, $j \geq k$, so that $b_1 = 0$, and consequently $d_1 = -a_1$. Furthermore, assuming for the moment that $j \geq k+1$, the $(1,1)$-entry of (4.3) entails the following relations between the terms of a_1 and c_0:

$$\begin{pmatrix} a_{1,k+1} \\ a_{1,k+2} \\ \vdots \\ a_{1,2j-2} \\ a_{1,2j-1} \end{pmatrix} + \begin{pmatrix} p_{1,j-1} & p_{1,j-2} & \cdots & p_{1,k-2+2} & p_{k-j+1} \\ 0 & p_{1,j-1} & \cdots & p_{1,k-j+3} & p_{1,k-j+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{1,j-1} & p_{1,j-2} \\ 0 & 0 & \cdots & 0 & p_{1,j-1} \end{pmatrix} \begin{pmatrix} c_{0,k+2} \\ c_{0,k+3} \\ \vdots \\ c_{0,2j-1} \\ c_{0,2j} \end{pmatrix} = 0 \ , \qquad (4.4)$$

and $a_{1,s} = 0$ for $s \geq 2j$.

On ℓ_2, the terms in (4.2) with u^2 are

$$\begin{pmatrix} \alpha_2 z^{2k} u^2 & \beta_2 z^{2k} u^2 \\ \gamma_2 z^{2k} u^2 & \delta_2 z^{2k} u^2 \end{pmatrix} =$$

$$\begin{pmatrix} a_2 u^2 + z^{-j}(p_2' \, c_0 + p_1 \, c_1 \, u) & b_2 \, u^2 \, z^{2j} + z^j \big((d_1 - a_1) \, u \, p_1 + (p_2 - p_2')\big) - c_0 \, p_1^2 \\ c_2 \, u^2 \, z^{-2j} & d_2 \, u^2 - z^{-j}(p_2 \, c_0 + p_1 \, c_1 \, u) \end{pmatrix} .$$

We need to examine the $(1,2)$-entry:

$$b_2 \, u^2 \, z^{2j} + z^j \big((d_1 - a_1) \, u \, p_1 + (p_2 - p_2')\big) - c_0 \, p_1^2 \qquad (4.5)$$

The conditions on the expression (4.5) is that all coefficients of $z^l \, u^2$ vanish for $l \geq 2k$, but any coefficient of $z^{2j} \, u^2$ and higher can be cancelled by choosing b_2 appropriately. So we only need to consider the range $k+1 \leq l \leq 2j-1$ to verify when the expression

$$z^j (d_1 - a_1) u \, p_1 + z^j (p_2 - p_2') - c_0 \, p_1^2$$

can be made holomorphic on V for any choice of p_2'. Given the determinant-one condition on the coordinate changes, this becomes

$$z^j (-2a_1) u \, p_1 + z^j (p_2 - p_2') - c_0 \, p_1^2 \ ,$$

and plugging in the values of p,

$$-2 z^j a_1 \sum_{s=k-j+1}^{j-1} p_{1s} \, z^s \, u + z^j \sum_{s=2k-j+1}^{j-1} \big(p_{2,s} - p_{2,s}'\big) z^s \, u^2 - c_0 \left(\sum_{s=k-j+1}^{j-1} p_{1s} \, z^s \, u \right)^2 . \qquad (4.6)$$

The terms to be cancelled are:

- Step 1, the coefficient of $z^{2k+1} u^2$:

$$\begin{aligned} &- 2\big(a_{1,k} p_{1,k-j+1} + a_{1,k-1} p_{1,k-j+2} + \cdots + a_{1,0} p_{1,2k-j+1}\big) + \big(p_{2,2k-j+1} - p_{2,2k-j+1}'\big) \\ &+ c_{0,2j-1} \, p_{1,k-j+1}^2 + 2 c_{0,2j-2} \, p_{1,k-j+1} \, p_{1,k-j+2} + c_{0,2j-3} \big(p_{1,k-j+2}^2 + 2 p_{1,k-j+1} \, p_{1,k-j+3}\big) \\ &+ \cdots + c_{0,1} \big(p_{1,k}^2 + 2 p_{1,k-1} \, p_{1,k+1} + \cdots\big) + c_{0,0} \big(2 p_{1,k} \, p_{1,k+1} + 2 p_{1,k-1} \, p_{1,k+2} + \cdots\big) = 0 \end{aligned}$$

- Step 2, the coefficient of $z^{2k+2}u^2$:

$$
\begin{aligned}
&-2\big(a_{1,k+1}p_{1,k-j+1}+a_{1,k}p_{1,k-j+2}+\cdots+a_{1,0}p_{1,2k-j+2}\big)+\big(p_{2,2k-j+2}-p'_{2,2k-j+2}\big)\\
&+c_{0,2j}\,p_{1,k-j+1}^2+2c_{0,2j-1}\,p_{1,k-j+1}\,p_{1,k-j+2}+c_{0,2j-2}\big(p_{1,k-j+2}^2+2p_{1,k-j+1}\,p_{1,k-j+3}\big)\\
&\qquad\qquad\qquad\qquad+\cdots+c_{0,0}\big(p_{1,k+1}^2+2p_{1,k}\,p_{1,k+2}+\cdots\big)=0
\end{aligned}
$$

- Step s, the coefficient of $z^{2k+s}u^2$ for $1\le s\le 2j-2k-1$, until...
- Step $2j-2k-1$, the coefficient of $z^{2j-1}u^2$:

$$
\begin{aligned}
&-2\big(a_{1,2j-k-2}p_{1,k-j+1}+a_{1,2j-k-3}p_{1,k-j+2}+\cdots+a_{1,0}p_{1,j-1}\big)+\big(p_{2,j-1}-p'_{2,j-1}\big)\\
&\quad+c_{0,2j}\big(2p_{1,-1}\,p_{1,0}+\cdots\big)+c_{0,2j-1}\big(p_{1,0}^2+2p_{1,-1}\,p_{1,1}+\cdots\big)+\cdots+c_{0,1}p_{1,j-1}^2=0
\end{aligned}
$$

Now *assume* $k\le j-1$, that is, assume that j is large. Then a term in $p_{1,0}^2$ appears with some $c_{0,s}$ in each of the above equations. Thus, choosing c_0 appropriately, we can solve them all, and we conclude that there are only restrictions on p'_2 when $p_{1,0}=0$. Now, we can carry out a similar argument for the coefficients $p_{1,s}$ for each $0\le s\le k$, so the set of non-generic bundles lives on the subvariety singled out by the equations

$$p_{1,0}=p_{1,1}=\cdots=p_{1,k}=0\ ,$$

thus having codimension at least $k+1$.

In the remaining case where $k=j$, we see directly from (2.3) that $p(z,u)=\sum_{s=1}^{k-1}p_{1s}\,z^s\,u$. Thus the only non-generic bundle is the split bundle, and the generic set of $\mathcal{M}_k(k)$ is precisely $\mathbb{P}^{k-2}$ when $k\ge 2$, and empty when $k=1$. (The same argument shows that the generic set is all of $\mathbb{P}^{2j-k-2}$ for $j\le k\le 2j-2$.) □

Theorem 4.12. *There is a topological embedding $\Phi\colon\mathcal{M}_j(k)\to\mathcal{M}_{j+k}(k)$. The image of Φ consists of all bundles in $\mathcal{M}_{j+k}(k)$ that split on the second infinitesimal neighbourhood of ℓ.*

Proof. Using the identification $\phi\colon\mathcal{M}_j(k)\to\mathbb{C}^m/{\sim}$, we define a map

$$\Phi\colon\mathcal{M}_j(k)\to\mathcal{M}_{j+k}(k)\quad\text{by}\quad(j,p)\mapsto(j+k,z^ku^2p)\ .$$

We want to show that Φ defines an embedding. We first show that the map is well defined: Suppose that $\begin{pmatrix}z^j & p\\ 0 & z^{-j}\end{pmatrix}$ and $\begin{pmatrix}z^j & p'\\ 0 & z^{-j}\end{pmatrix}$ represent isomorphic bundles. Then there are coordinate changes $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ holomorphic in (z,u) and $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ holomorphic in (z^{-1},z^ku) such that

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}z^j & p'\\ 0 & z^{-j}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}z^{-j} & -p\\ 0 & z^j\end{pmatrix}\ .$$

Therefore these two bundles are isomorphic exactly when the system of equations

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}a+z^{-j}p'c & z^{2j}b+z^j(p'd-ap)-pp'c\\ z^{-2j}c & d-z^{-j}pc\end{pmatrix}\tag{4.7}$$

can be solved by a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ holomorphic in (z,u) which makes $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ holomorphic in $(z^{-1}, z^k u)$.

On the other hand, the images of these two bundles are given by transition matrices $\left(\begin{smallmatrix} z^{j+k} & z^k u^2 p \\ 0 & z^{-j-k} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} z^{j+k} & z^k u^2 p' \\ 0 & z^{-j-k} \end{smallmatrix}\right)$, which represent isomorphic bundles if and only if there are coordinate changes $\left(\begin{smallmatrix} \bar a & \bar b \\ \bar c & \bar d \end{smallmatrix}\right)$ holomorphic in (z,u) and $\left(\begin{smallmatrix} \bar\alpha & \bar\beta \\ \bar\gamma & \bar\delta \end{smallmatrix}\right)$ holomorphic in $(z^{-1}, z^k u)$ satisfying the equality

$$\begin{pmatrix} \bar\alpha & \bar\beta \\ \bar\gamma & \bar\delta \end{pmatrix} = \begin{pmatrix} z^{j+k} & z^k u^2 p' \\ 0 & z^{-j-k} \end{pmatrix} \begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix} \begin{pmatrix} z^{-j-k} & -z^k u^2 p \\ 0 & z^{j+k} \end{pmatrix} .$$

That is, the images represent isomorphic bundles if the system

$$\begin{pmatrix} \bar\alpha & \bar\beta \\ \bar\gamma & \bar\delta \end{pmatrix} = \begin{pmatrix} \bar a + z^{-j} u^2 p' \bar c & z^{2k}\bigl(z^{2j}\bar b + z^j u^2 (p'\bar d - \bar a p) - u^4 p p' \bar c\bigr) \\ z^{-2j-2k}\bar c & \bar d - z^{-j} u^2 p \bar c \end{pmatrix} \tag{4.8}$$

has a solution.

Write $x = \sum x_i u^i$ for $x \in \{a, b, c, d, \bar a, \bar b, \bar c, \bar d\}$ and choose

$$\bar a_i = a_{i+2k} \ , \quad \bar b_i = b_{i+2k} u^2 \ , \quad \bar c_i = c_{i+2k} u^{-2} \ , \quad \bar d_i = d_{i+2k} \ .$$

Then if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ solves (4.7), one verifies that $\left(\begin{smallmatrix} \bar a & \bar b \\ \bar c & \bar d \end{smallmatrix}\right)$ solves (4.8), which implies that the images represent isomorphic bundles and therefore ϕ is well defined. To show that the map is injective, just reverse the previous argument. Continuity is obvious. Now we observe also that the image $\phi(\mathcal{M}_j)$ is a saturated set in $\mathcal{M}_{j+k}$, meaning that if $y \sim x$ and $x \in \phi(\mathcal{M}_j)$ then $y \in \phi(\mathcal{M}_j)$. In fact, if $E \in \phi(\mathcal{M}_j)$, then E splits on the second infinitesimal neighbourhood. Now if $E' \sim E$, then E' must also split on the second infinitesimal neighbourhood, and therefore the polynomial corresponding to E' is of the form $u^2 p'$, and hence $\phi(z^{-k}p')$ gives E'. Note also that $\phi(\mathcal{M}_j)$ is a closed subset of $\mathcal{M}_{j+k}$, given by the equations $p_{il} = 0$ for $i = 1, 2$. Now the fact that ϕ is a homeomorphism over its image follows from the easy Lemma 4.14 given below. $\square$

Remark 4.13. R. Moraru gave us the following coordinate-free expression of the embedding map $\Phi \colon \mathcal{M}_j(k) \to \mathcal{M}_{j+k}(k)$:

$$\Phi\bigl(E\bigr) = \otimes\, \mathcal{O}(-k) \circ \mathrm{Elm}_{\mathcal{O}_\ell(j+k)} \circ \mathrm{Elm}_{\mathcal{O}_\ell(j)}\bigl(E\bigr) \ ,$$

where Elm_L denotes the elementary transformation with respect to the line bundle L. Using this coordinate-free expression it becomes obvious that Φ is well defined.

Lemma 4.14. *Let $X \subset Y$ be a closed subset and $\sim$ an equivalence relation in Y such that X is $\sim$-saturated. Then the map $I \colon X/\!\sim\, \to Y/\!\sim$ induced by the inclusion is a homeomorphism over the image.*

Proof. Denote the projections by $\pi_X \colon X \to X/\!\sim$ and $\pi_Y \colon Y \to Y/\!\sim$. Let F be a closed subset of $X/\!\sim$. Then $\pi_X^{-1}(F)$ is closed and saturated in X, and therefore $\pi_X^{-1}(F)$ is also closed and saturated in Y. It follows that $\pi_Y\bigl(\pi_X^{-1}(F)\bigr)$ is closed in $Y/\!\sim$. $\square$

Theorem 4.15. *If $j = nk$ for some $n \in \mathbb{N}$, then the pair $(\mathbf{h}_k, \mathbf{w}_k)$ stratifies instanton moduli stacks $\mathcal{M}_j(k)$ into Hausdorff components.*

Proof. This proof uses the same techniques as that of [BG1, Theorem 4.1]. On the first infinitesimal neighbourhood, we have two possibilities: in the first case we have bundles belonging to the open dense subset $\mathbb{P}^{2j-2-k}$, singled out by having the lowest possible values of numerical invariants (see Remark 4.16 for a question about stability); in the second case, at least one of the invariants is strictly higher than the lower bound, and such bundles are separated away from the most generic stratum. On the second infinitesimal neighbourhood, the problem is solved by first separating the most generic stratum from the other ones. For the remaining part of the second neighbourhood, one divides the polynomial by u falling back to the same analysis done for the first neighbourhood. We are then left only with bundles which split on the second neighbourhood. We use induction j, assuming that the invariants stratify $\mathcal{M}_{j-k}(k)$ into Hausdorff components together with the embedding Theorem 4.12, stating that $\Phi\big(\mathcal{M}_{j-k}(k)\big)$ is the set of bundles on $\mathcal{M}_j(k)$ that split on the second infinitesimal neighbourhood. $\square$

Remark 4.16. When this paper was nearly completed, we noticed the need to define a notion of stability on $\mathcal{M}_k(j)$ in order to have full compatibility between the methods used in the proof of Theorem 4.11 and the stratification presented in Theorem 4.15. Defining stability for a bundle E via counting the dimension of $H^1(Z_k;\ \mathscr{E}nd\, E)$ seemed to us to be the most natural choice, and had the pleasant feature of being a notion that fits extremely well with the standard deformation theory that is well known for the compact case; nevertheless, it was just an ad-hoc definition of stability. Since there are many inequivalent ways to define stability, we chose to postpone this question to a future paper.

Example 4.17. $\mathcal{M}_3(1)$ is the simplest example in which the local holomorphic Euler characteristic $\chi(\ell, E)$ does not distinguish Hausdorff components: Writing $\chi = \mathbf{w}_1 + \mathbf{h}_1$, the (Hausdorff) generic set has bundles with $\chi = 3 = 1 + 2$, but the set of bundles with $\chi = 5$ is non-Hausdorff, containing both bundles of characteristic $\chi = 3 + 2$ and $\chi = 2 + 3$. It is necessary to fix both $\mathbf{w}_1$ and $\mathbf{h}_1$ to obtain Hausdorff subspaces of $\mathcal{M}_3(1)$.

Example 4.18. Bundles on Z_2 with splitting type $j = 3$ do not represent instantons. The extension class is represented by the polynomial $p(z, u) = (p_{10} + p_{11}z + p_{12}z^2)u + p_{22}z^2u^2$. The generic set of $\mathcal{M}_3(2)$ is a projective 2-space given by $[p_{10} : p_{11} : p_{12}]$ minus the subvariety $p_{10} = p_{12} = 0$. Generic bundles have invariants (width, height)$= (0, 2)$, while the non-generic bundles given by u and z^2u have invariants $(1, 2)$, and the bundle given by z^2u^2 has invariants $(2, 2)$, *like the split bundle represented by* $p = 0$. Thus the invariants do not separate $\mathcal{M}_3(2)$ into Hausdorff components (the split bundle is never separated from any other bundle), which contrasts the situation for instanton moduli $\mathcal{M}_{nk}(k)$.

References

[BG1] Ballico, E. and Gasparim, E. *Numerical invariants for vector bundles on blow-ups*, Proc. Amer. Math. Soc. **130** (2002) 23–32.

[BG2] Ballico, E. and Gasparim, E. *Vector bundles on a neighborhood of an exceptional curve and elementary transformations*, Forum Math. **15** (2003) 115–122.

[BG3] Ballico, E. and Gasparim, E. *Vector bundles on a formal neighborhood of a curve in a surface*, Rocky Mountain J. Math. **30** (2000) 795–814.

[Bl] Blache, R. *Chern classes and Hirzebruch–Riemann–Roch theorem for coherent sheaves on complex projective orbifolds with isolated singularities*, Math. Z. **222** (1996) 7–57.

[Ei] Eisenbud, D. *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** no. 1 (1980) 35–64.

[Ga1] Gasparim, E. *Holomorphic bundles on $\mathcal{O}(-k)$ are algebraic*, Comm. Algebra **25** (1997) 3001–3009.

[Ga2] Gasparim, E. *Rank two bundles on the blow-up of* $\mathbb{C}^2$, J. Algebra **199** (1998) 581–590.

[Ga3] Gasparim, E. *Chern classes of bundles on blown-up surfaces*, Comm. Algebra **28** (2000) 4912–4926.

[Ga4] Gasparim, E. and Ontaneda, P. *Three applications of instanton numbers*, Comm. Math. Phys. **270** no. 1 (2007) 1–12.

[GKM] Gasparim, E., Köppe, T. and Majumdar, P. *Local holomorphic Euler characteristic and instanton decay*, Pure Appl. Math. Q. **4** no. 2, Special Issue: In honor of Fedya Bogomolov, Part 1 (2008) 161–179.

[M2] Grayson, D. R. and Stillman, M. E. *Macaulay 2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.

[Gr] Griffiths, P. A. *The extension problem in complex analysis, II. Embeddings with positive normal bundle*, Amer. J. Math. **88** (1966) 366–446.

[GS] Gasparim, E. and Swanson, I. *Computing Instanton numbers of curve singularities*, J. Symbolic Computation **40** no. 2 (2005) 965–978

[Ha] Hartshorne, R. *Algebraic Geometry*, Graduate Texts in Mathematics, Springer Verlag (1977).

[LT] Lübke, M. and Teleman, A. *The Kobayashi–Hitchin correspondence*, World Scientific Publishing Co. Inc., River Edge, NJ (1995).

[Re] Reid, M. *La Correspondance de McKay*, Séminaire Bourbaki, 52ème année, novembre 1999, no. 867, to appear in Astérisque 2000.

[So] Souza, A. F. P. *Fibrados sobre* $\mathcal{K}_{\mathbb{P}^1}$, Master's thesis UFPE (1999) (Portuguese).

Edoardo Ballico
University of Trento, Department of Mathematics

I–38050 Povo (Trento), Italia
E-mail: ballico@science.unitn.it

Elizabeth Gasparim and Thomas Köppe
School of Mathematics, The University of Edinburgh
James Clerk Maxwell Building, The King's Buildings, Mayfield Road
Edinburgh, EH9 3JZ, United Kingdom
E-mail: Elizabeth.Gasparim@ed.ac.uk
E-mail: t.koeppe@ed.ac.uk