Smoothing of rational m-ropes

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Abstract

In a recent paper, Gallego, González and Purnaprajna showed that rational 3-ropes can be smoothed. We generalise their proof, and obtain smoothability of rational m-ropes for $m \geq 3$.

1 Introduction

We generalise the smoothing theorem for rational 3-ropes of Gallego, González and Purnaprajna to rational m-ropes with $m \geq 3$. Our proof uses their construction presented in [9].

Let C be a smooth, irreducible projective curve. A *rope* X of multiplicity $m \geq 2$ over C is a nowhere reduced scheme X whose reduced structure is C and which locally looks like the first infinitesimal neighborhood of C inside the total space of a vector bundle of rank $m-1$ ([6, 9]).

Since the ideal sheaf $\mathcal{E} := \mathcal{I}_{C,X}$ of C inside X has square zero, it may be seen as a coherent $\mathcal{O}_C$-sheaf, the so-called conormal bundle or conormal module of C. As an $\mathcal{O}_C$-sheaf $\mathcal{E}$ is locally free of rank $m-1$.

Our goal is to show smoothability of rational m-ropes. We recall the precise definitions:

Definition 1.1. Let Y be a reduced connected scheme and let $\mathcal{E}$ be a locally free sheaf of rank $m-1$ on Y. A *rope of multiplicity* m or m*-rope* on Y with conormal bundle $\mathcal{E}$ is a scheme X with $X_{\mathrm{red}} = Y$ such that

- $\mathcal{I}_{Y,X}^2 = 0$ and
- $\mathcal{I}_{Y,X}|_Y \cong \mathcal{E}$ as $\mathcal{O}_Y$-modules.

Definition 1.2. A *smoothing of a rope* X is a flat integral family $\mathcal{X}$ of schemes over a smooth affine curve T such that over a point $0 \in T$ we have $\mathcal{X}_0 = X$, and X_t is a smooth irreducible variety over the remaining points $t \in T \setminus \{0\}$.

Here we consider only the case when $Y = C$ is a smooth curve with arithmetic genus $q := p_a(C) = 1 - \chi(\mathcal{O}_X)$, and we work over an algebraically closed field of characteristic zero. Any m-rope X on C with conormal module $\mathcal{E}$ gives an extension class

$$\epsilon \in \mathrm{Ext}^1_{\mathcal{O}_X}\big(\omega_C, \mathcal{E}\big) \cong H^1\big(C;\ \mathcal{E} \otimes \omega_C^*\big)$$

(cf. [10, 1.2] or [5, §1] for the case $m = 2$). Two ropes X, X' with conormal module $\mathcal{E}$ are isomorphic over Y if and only if their extension classes are in the same orbit by the action of $\mathrm{Aut}(\mathcal{E})$ on $\mathrm{Ext}^1_{\mathcal{O}_X}\big(\omega_C, \mathcal{E}\big)$ (cf. [10, 1.2]). There is an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \to 0\ , \tag{1}$$

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and so $\chi(\mathcal{O}_X) = m \cdot \chi(\mathcal{O}_C) + \deg(\mathcal{E}) = m(1-q) + \deg(\mathcal{E})$. Let $g := p_a(X) = 1 - \chi(\mathcal{O}_X)$ be the arithmetic genus of X. Obviously, if X is a flat limit of a family of smooth, connected projective curves, then $\chi(\mathcal{O}_X) \leq 1$, i.e. $\deg(\mathcal{E}) \leq m(q-1)+1$, and g is the genus of the nearby smooth curves. Hence $g \geq 0$. Here (as in [9, §4]) we will only consider *rational* m-ropes, i.e. we will assume that $q = 0$. In the case $m = 3$ Gallego, González and Purnaprajna proved that if $p_a(X) \geq 0$, then the rational 3-rope X may be smoothed, both as abstract scheme and as scheme embedded in a fixed projective space [9, Theorem 4.5]. Here we use their proof to solve the case $m \geq 4$, proving the following result.

Theorem 1.3 (Main theorem)**.** *Fix integers $r > m \geq 3$ and $g \geq 0$ and let X be any rational m-rope such that $1 - \chi(\mathcal{O}_X) = g$. Then X may be smoothed as an abstract scheme.*

Moreover, there exist an embedding $j\colon X \to \mathbb{P}^r$ and a flat family $\{\mathcal{X}_t\}_{t\in T}$ of subschemes of $\mathbb{P}^r$ parametrized by an integral and smooth affine curve T with the following properties:

1. *There exists a point $0 \in T$ such that $j(X) = \mathcal{X}_0$, and*
2. *for all $t \in T\backslash\{0\}$, $\mathcal{X}_t$ is a smooth connected curve of genus g and degree $m \cdot \deg j(X)$.*

For the existence of embedding $j\colon X \to \mathbb{P}^r$ with a fixed degree, see Lemma 2.6 and Remark 2.7.

2 Proof of the Main Theorem

We begin by collecting a few results which show the existence of many non-degenerate embeddings of an m-rope in $\mathbb{P}^r$ for all $r \geq m+1$.

Lemma 2.1. *Fix an integer $m \geq 3$ and let E be a vector bundle of rank $m-1$ on $\mathbb{P}^1$. There is a uniquely determined sequence of integers $b_1 \geq \cdots \geq b_{m-1}$ such that $E \cong \bigoplus_{i=1}^{m-1} \mathcal{O}_{\mathbb{P}^1}(b_i)$, and $\deg(E) = b_1 + \cdots + b_{m-1}$. Then E is rigid if and only if $b_1 \leq b_{m-1}+1$.*

Proof. This is a classical result of the deformation theory of vector bundles on $\mathbb{P}^1$. □

Let X be an m-rope over C with canonical module $\mathcal{E} = \mathcal{I}_{C,X}$. Since $\mathcal{E}^2 = 0$ and $H^2(C;\ \mathcal{E}) = 0$, there is an exact sequence of Abelian groups

$$0 \longrightarrow H^1(C;\ \mathcal{E}) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(C) \longrightarrow 1\ , \tag{2}$$

in which the group structure of $H^1(C;\ \mathcal{E})$ as a subgroup of $\mathrm{Pic}(X)$ is the usual addition of the $\daleth$-vector space (see [11, p. 446] for the case $\daleth = \mathbb{C}$, or the proof of [5, Proposition 4.1] for an arbitrary field $\daleth$). Hence for every $L \in \mathrm{Pic}(C)$ there exists an $L' \in \mathrm{Pic}(X)$ such that $L'|_C \cong L$. Now assume that $q := p_a(C) = 0$, so $C \cong \mathbb{P}^1$. By Lemma 2.1 there are integers $a_1 \geq \cdots \geq a_{m-1}$ such that $\mathcal{E} \cong \bigoplus_{i=1}^{m-1} \mathcal{O}_{\mathbb{P}^1}(-a_i)$.

Lemma 2.2. *With the notation as above, fix $d \in \mathbb{Z}$ and some $L_d \in \mathrm{Pic}(X)$ such that $L_d|_C \cong \mathcal{O}_{\mathbb{P}^1}(d)$. If $d \geq \max\{2, a_1+1\}$, then L_d is very ample, $h^1(X;\ L_d) = 0$, and $h^0(X;\ L_d) = (m+1)d - \sum_{i=1}^{m-1} a_i$.*

Proof. The last assertion is obvious, because $h^1\big(\mathbb{P}^1;\ \mathcal{O}_{\mathbb{P}^1}(d)\big) = h^1\big(\mathbb{P}^1;\ \mathcal{E}(d)\big) = 0$. To check the very ampleness of L_d, it suffices to prove that $h^0\big(X;\ \mathcal{I}_Z \otimes L_d\big) = h^0\big(X;\ L_d\big) - 2$ (or, equivalently, $h^1\big(X;\ \mathcal{I}_Z \otimes L_d\big) = 0$) for any length-2 zero-dimensional subscheme $Z \subset X$. Fix an affine neighborhood U of Z in X. Since every affine m-rope is split, there is a retraction $u\colon U \to U \cap C$. There is a length-2 zero-dimensional scheme W such that $Z \subset u^{-1}(W)$. Hence it is sufficient to prove that $h^1\big(X;\ \mathcal{I}_{u^{-1}(W)} \otimes L_d\big) = 0$. The latter vanishing is true, because $h^1\big(\mathbb{P}^1;\ \mathcal{O}_{\mathbb{P}^1}(d-2)\big) = h^1\big(\mathbb{P}^1;\ \mathcal{E}(d-2)\big) = 0$. Twisting (1) with $\mathcal{O}_{\mathbb{P}^1}(d)$ we get $h^1(X;\ L_d) = 0$, and $h^0(X;\ L_d) = (m+1)d - \sum_{i=1}^{m-1} a_i$. □

Taking d as in the proof of Theorem 1.3 below, we see that in general we are able to smooth only certain types of embeddings.

Lemma 2.3. *Let Y be a smooth curve of genus g and $m \in \mathbb{Z}$ such that $m \geq \max\{g+1, 2\}$. Let R be a general element in $\mathrm{Pic}^m(Y)$. There exists a general two-dimensional linear subspace V of $H^0(Y;\ R)$ that spans R, and any such V determines a degree-m morphism $f\colon Y \to \mathbb{P}^1$. Then the sheaf $G := f_*(\mathcal{O}_Y)/\mathcal{O}_{\mathbb{P}^1}$ is locally free of rank $m-1$, and G is rigid.*

Proof. Since R is general and $m \geq g+1$, $h^1(Y;\ R) = 0$. Thus, $h^0(Y;\ R) = m+1-g \geq 2$ by Riemann-Roch. The generality of R implies that R is spanned, and hence a general two-dimensional linear subspace V of $H^0(Y; R)$ spans R. Any such V determines a degree-m morphism $f\colon Y \to \mathbb{P}^1$. Since $\mathcal{O}_Y$ is torsion-free, so is $f_*(\mathcal{O}_Y)$, which is therefore locally free; also $h^0\big(\mathbb{P}^1;\ f_*(\mathcal{O}_Y)\big) = h^0\big(Y;\ \mathcal{O}_Y\big) = 1$. Therefore $f_*(\mathcal{O}_Y)$ has precisely one trivial line subbundle, so the sheaf $G := f_*(\mathcal{O}_Y)/\mathcal{O}_{\mathbb{P}^1}$ is locally free. Let $b_1 \geq \cdots \geq b_{m-1}$ be the splitting type of G, so $b_1 + \cdots + b_{m-1} = \deg(G)$. Since $1-g = \chi(\mathcal{O}_Y) = \chi(G) + \chi(\mathcal{O}_{\mathbb{P}^1}) = \deg(G) + m$, we get $\deg(G) = 1-m-g$. Since $h^0(Y;\ \mathcal{O}_Y) = h^0(\mathbb{P}^1;\ \mathcal{O}_{\mathbb{P}^1}) = 1$, $h^0(\mathbb{P}^1; G) = 0$, i.e. $b_1 < 0$. Since $R \cong f^*(\mathcal{O}_{\mathbb{P}^1}(1))$, we have $h^1\big(Y;\ R\big) = h^1\big(\mathbb{P}^1;\ G(1)\big)$ by the projection formula. Since $h^1(Y; R) = 0$, we get $b_{m-1}+1 \geq -1$. Hence $b_{m-1} \geq b_1 - 1$, and G is rigid by Lemma 2.1. □

Lemma 2.4. *Fix integers m, g such that $2 \leq m \leq g \leq 2m-2$ and let Y be a general smooth curve with genus g. There exists a line bundle $R \in \mathrm{Pic}^m(Y)$ such that $h^0(Y;\ R) = 2$ and R is spanned. Hence R determines a degree-m morphism $f\colon Y \to \mathbb{P}^1$ such that $R \cong f^*\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$. Then the sheaf $G := f_*(\mathcal{O}_Y)/\mathcal{O}_{\mathbb{P}^1}$ is locally free of rank $m-1$, and G is rigid.*

Proof. Brill-Noether theory gives the existence of $R \in \mathrm{Pic}^m(Y)$ such that $h^0(Y;\ R) = 2$ and R is spanned [2, Theorem V.1.1]. The sheaf G is locally free by the same argument as in the proof of Lemma 2.3. Let $b_1 \geq \cdots \geq b_{m-1}$ be the splitting type of G. As in the proof of Lemma 2.3, the projection formula gives, for all $c \in \mathbb{Z}_{\geq 0}$,

$$h^0\big(Y;\ R^{\otimes c}\big) = h^0\big(\mathbb{P}^1;\ \mathcal{O}_{\mathbb{P}^1}(c)\big) + h^0\big(\mathbb{P}^1;\ G(c)\big)\ ,$$

i.e. $h^0\big(\mathbb{P}^1;\ G(c)\big) = h^0\big(Y;\ R^{\otimes c}\big) - c - 1$. Since $h^0(Y;\ R) = 2$, we get $h^0\big(Y;\ G(1)\big) = 0$, i.e. $b_1 \leq -2$. The Gieseker-Petri theorem gives $h^1\big(Y;\ R^{\otimes 2}\big) = 0$ [1, Cor. 5.7]. Hence $b_{m-1} \geq -3$, and G is rigid by Lemma 2.1. □

Lemma 2.5. *Let $D \subset \mathbb{P}^r$ be a smooth rational curve of degree $d > 0$ and assume that $r \geq 2$. Let $\mathcal{N}_D$ be the normal bundle of D in $\mathbb{P}^r$ and $n_1 \geq \cdots \geq n_{r-1}$ its splitting type. Then $n_{r-1} \geq d$.*

Proof. The Euler sequence of $T\mathbb{P}^r$ shows that $T\mathbb{P}^r(-1)$ is spanned. Consequently, $T\mathbb{P}^r(-1)|_D$ is spanned. Since D is a closed submanifold of $\mathbb{P}^r$, there is a surjection $T\mathbb{P}^r|_D \to \mathcal{N}_D$. Thus, $\mathcal{N}_D(-1)$ is spanned, i.e. $n_{r-1} - d \geq 0$. □

Lemma 2.6. *Fix integers $r > m \geq 2$, $d > 0$, let $\mathcal{E}$ be a vector bundle of rank $m-1$ on $\mathbb{P}^1$ of splitting type $e_1 \geq \cdots \geq e_{m-1}$, and let X be the rational m-rope with conormal bundle $\mathcal{E}$.*

Fix any embedding $u\colon \mathbb{P}^1 \to \mathbb{P}^r$ (we do not assume that $u(\mathbb{P}^1)$ spans $\mathbb{P}^r$) and let $d := \deg u(\mathbb{P}^1)$. If $d \geq -e_{m-1}$, then there exists an embedding $j\colon X \to \mathbb{P}^r$ such that $j|_{X_{\mathrm{red}}} = u$. Moreover, $h^1\big(\mathbb{P}^1;\ \mathcal{E} \otimes u^*\mathcal{O}_{\mathbb{P}^r}(1)\big) = 0$.

Proof. Set $D := u(\mathbb{P}^1)$ and let $\mathcal{N}_D$ be the normal bundle of D in $\mathbb{P}^r$. Let $n_1 \geq \cdots \geq n_{r-1}$ be the splitting type of $\mathcal{N}_D$; so the splitting type of $\mathcal{N}_D^*$ is $-n_{r-1} \geq \cdots \geq -n_1$. By Lemma 2.5 we have $-n_i \leq -d$ for all i. By [10, Proposition 2.1] or [9, Theorem 2.2], there is a one-to-one correspondence between the surjections $\mathcal{N}_D^* \to \mathcal{E}$ and embeddings $j\colon X \to \mathbb{P}^r$ such that $j|_{X_{\mathrm{red}}} = u$. Since $\operatorname{rk} \mathcal{N}_D^* = r-1 > \operatorname{rk} \mathcal{E}$, a surjection $\mathcal{N}_D^* \to \mathcal{E}$ exists if $-d \leq e_{m-1}$, i.e. if $d \geq -e_{m-1}$. The last sentence is obvious, because $h^1\big(\mathbb{P}^1;\ \mathcal{E}(t)\big) = 0$ if and only if $t \geq -e_{m-1} - 1$. □

As an aside, the following observation shows the existence of many rational m-ropes in $\mathbb{P}^m$, but notice that their conormal bundles must satisfy very strong restrictions. Since in the statement of Theorem 1.3 we assume $r > m$, these are not the ropes that our theorem addresses.

Remark 2.7 (Embedding m-ropes in $\mathbb{P}^m$)**.** Fix integers $m \geq 2$, $d > 0$ and a vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ of rank $m-1$ with splitting type $e_1 \geq \cdots \geq e_{m-1}$. Let X be the rational m-rope with conormal bundle $\mathcal{E}$. Let $u\colon \mathbb{P}^1 \to \mathbb{P}^m$ be an embedding such that the curve $D := u(\mathbb{P}^1)$ has degree d. Let $\mathcal{N}_D$ be the normal bundle of D in $\mathbb{P}^r$ and $n_1 \geq \cdots \geq n_{m-1}$ its splitting type. Since $\operatorname{rk} \mathcal{E} = \operatorname{rk} \mathcal{N}_D$, any surjection $\mathcal{N}_D^* \to \mathcal{E}$ must be an isomorphism.

Hence there exists an embedding $j\colon X \to \mathbb{P}^m$ such that $j|_{X_{\mathrm{red}}} = u$ if and only if $\mathcal{N}_D^* \cong \mathcal{E}$ [10, Proposition 2.1 (2)]. Thus, the existence problem of embeddings j of X such that $j|_{X_{\mathrm{red}}}$ is associated to a subseries of $H^0\big(\mathbb{P}^1;\ \mathcal{O}_{\mathbb{P}^1}(d)\big)$, and is equivalent to the study of all possible splitting types of the normal bundles $\mathcal{N}_D$ for some $D = u(\mathbb{P}^1)$.

The case $m = 2$ is trivial, because we must have $1 \leq d \leq 2$, so D is either a line or a smooth conic. From now on we assume $m \geq 3$. We first consider the embeddings spanning $\mathbb{P}^m$. Thus we assume for a moment $d \geq m$. If $m = 3$, then the set of all splitting types arising in this way is known, and the set of all smooth rational space-curves with fixed normal bundle has a very interesting geometry [8]. If $m > 3$, then all possible splitting types $n_1 \geq \cdots \geq n_{m-1}$ that may arise if we allow the map $\mathbb{P}^1 \to \mathbb{P}^r$ to be unramified but not necessarily injective are described in [13]. For arbitrary m, the rigid vector bundle, i.e. the one with $b_{m-1} \geq b_1 - 1$, arises as the normal bundle of the general degree-d embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^m$.

Now we look at the embeddings for which D spans a k-dimensional linear subspace M of $\mathbb{P}^m$ for some $k < m$. Let $\mathcal{N}_{D,M}$ denote the normal bundle of D in M with splitting type $b_1 \geq \cdots \geq b_{k-1}$. By Lemma 2.5 we have $b_{k-1} \geq d$. Since $\mathcal{N}_D \cong \mathcal{N}_{D,M} \oplus \mathcal{O}_D(1)^{\oplus(m-k)}$, we get $n_i = b_i$ if $1 \leq i \leq k-1$ and $n_i = d$ if $k \leq i \leq m-1$. Assume that $\mathcal{E}^* \cong \mathcal{N}_D$, i.e. assume the existence of degree-d embedding u of $\mathbb{P}^1$ and embedding $j\colon X \to \mathbb{P}^m$ such that $j|_{X_{\mathrm{red}}} = u$. By Lemma 2.5 we have $e_1 \leq -d$. If $u(\mathbb{P}^1)$ spans $\mathbb{P}^m$, then we have $e_1 \leq -d-1$. Since $\deg \mathcal{N}_D = (m+1)d - 2$, we have $e_1 + \cdots + e_{m-1} = 2 - (m+1)d$. According to [13], these are the only restrictions if we allow unramified but non-injective maps u. Notice that $h^1\big(\mathbb{P}^1;\ \mathcal{E} \otimes u^*\mathcal{O}_{\mathbb{P}^r}(1)\big) = 0$ if and only if $d \geq -e_{m-1} - 1$. If $u(\mathbb{P}^1)$ spans $\mathbb{P}^m$, then this condition is satisfied only if $e_1 = e_{m-1}$, i.e. if and only if $\mathcal{E}$ is balanced. //

We are now in a position to prove the main theorem.

Proof of Theorem 1.3. Let $\mathcal{E}$ be the conormal bundle of the m-rope X, and let $e_1 \geq \cdots \geq e_{m-1}$ be the splitting type of $\mathcal{E}$. Also, let G be the only rigid vector bundle on $\mathbb{P}^1$ with rank $m-1$ and degree $1-g-m$. If $g \geq m$, then there exists a degree-m covering $f\colon Y \to \mathbb{P}^1$ such that Y is a smooth curve of genus g and $f_*(\mathcal{O}_Y) \cong \mathcal{O}_{\mathbb{P}^1} \oplus G$ [3, Proposition 1].

There are various ways to see the equivalence between the rigidity of G and the statements in [3] or in [7, Proposition 2.1.1]. We can just use our Lemma 2.4; alternatively the reader may wish to consult [13, 2.4, 2.5] or [4, Remark 1]. For a different proof in the case $g \geq 2m+1$, see [7, Proposition 2.1.1]; there Y is a general m-gonal curve of genus g.

Lemma 2.3 gives the same result if $m \geq \max\{g+1, 2\}$, and Lemma 2.4 shows that such coverings are very common. Hence for all integers $g \geq 0$ there is a smooth curve Y of genus g and a degree-m morphism $f\colon Y \to \mathbb{P}^1$ such that the rank-$(m-1)$ vector bundle $f_*(\mathcal{O}_Y)/\mathcal{O}_{\mathbb{P}^1}$ is rigid.

By [9, Corollary 2.7], any rational m-rope with conormal bundle G may be smoothed. Moreover, for any $r > m$, Lemma 2.6 yields the existence of an embedding $j\colon X \hookrightarrow \mathbb{P}^r$, where $j^*\mathcal{O}_{\mathbb{P}^r}(1) = L_n$ for some suitable n. (We also get the existence of such an embedding for $r = m$ from Remark 2.7.) Recall that $L_n|_C \cong \mathcal{O}_{\mathbb{P}^1}(n)$ and that $H^1\big(C;\ \mathcal{E} \otimes L_n|_C\big) = H^1\big(C;\ L_n|_C\big) = 0$. Then we may apply [9, Theorem 2.4], and $j(X)$ can be smoothed inside $\mathbb{P}^r$.

Note that since G is rigid, the condition $h^1\big(\mathbb{P}^1;\ G \otimes j|_{X_{\mathrm{red}}}^*(\mathcal{O}_{\mathbb{P}^r}(1))\big) = 0$ is satisfied if and only if $j|_{X_{\mathrm{red}}}$ is a degree-d embedding of $\mathbb{P}^1$ such that $d(m-1)+1-g-m \geq 1-m$, i.e. if and only if $d(m-1) \geq g$. Any vector bundle E on $\mathbb{P}^1$ of rank $m-1$ and degree $1-g-m$ is a degeneration of a flat family of vector bundles on $\mathbb{P}^1$ isomorphic to G. To get an embedding of X, we need a degree-d embedding of $\mathbb{P}^1$ with $d \geq -e_{m-1}$. If X has E as conormal bundle, we need to assume that $\deg j(X_{\mathrm{red}}) \geq -e_{m-1}-1$, where now $e_1 \geq \cdots \geq e_{m-1}$ is the splitting type of E. Notice that we cannot fix the same integer $\deg j(X_{\mathrm{red}})$ for all bundles E with rank $m-1$ and degree $1-g-m$. Very nice degeneration techniques are given in [9, Propositions 4.3 and 4.4], and Case 2 of the proof of Theorem 4.5 shows that the smoothing (both as abstract schemes and as embedded schemes) is true for arbitrary rational m-ropes with the same arithmetic genus g, if we only consider as their supports embeddings $u\colon \mathbb{P}^1 \to \mathbb{P}^r$ such that $\deg u(\mathbb{P}^1) \geq -e_{m-1}-1$. The key condition $h^1\big(\mathbb{P}^1;\ B \otimes j^*(\mathcal{O}_{\mathbb{P}^r}(1))\big) = 0$ both for $B = E$ and for $B = G$ is satisfied by the last sentence of Lemma 2.6.

For reader's sake we summarize the part of the proof in [9] that we need: Let X be a rational m-rope with arithmetic genus g. Since E is a degeneration of G, there are an integral scheme S, $0 \in S$, and a rank-$(m-1)$ vector bundle $\mathbb{E}$ on $\mathbb{P}^1 \times S$ such that

$$\mathbb{E}|_{p_2^{-1}(0)} \cong E \quad \text{and} \quad \mathbb{E}|_{p_2^{-1}(s)} \cong G \ \text{ for a general } s \in S \setminus \{0\},$$

where $p_2\colon \mathbb{P}^1 \times S \to S$ is the projection onto the second factor. Let $p_1\colon \mathbb{P}^1 \times S \to \mathbb{P}^1$ be the projection on the first factor. Set $\mathcal{A} := \mathcal{E}xt^1_{p_2}\big(p_1^*(\omega_{\mathbb{P}^1}), \mathbb{E}\big)$, where $\mathcal{E}xt^1_{p_2}$ is the relative $\mathcal{E}xt^1$-sheaf with respect to p_2. The $\mathcal{O}_S$-sheaf $\mathcal{A}$ is coherent. If the splitting type of E is very unbalanced (i.e. if it contains an integer ≥ -2), then $\mathcal{A}$ is not locally free. The total space $\mathbb{V}(\mathcal{A})$ of $\mathcal{A}$ parametrizes a family of rational m-ropes containing all m-ropes with conormal bundle E and all m-ropes with conormal bundle G. Notice that "smoothing " is a closed condition. When $\mathcal{A}$ is locally free, then $\mathbb{V}(\mathcal{A})$ is irreducible and the rope X is smoothable. To handle the general case, Gallego, González and Purnaprajna made the following nice observation [9, Proof of Proposition 4.4]: Even if $\mathcal{A}$ is not locally free, the fact that the fibers of p_2 have dimension 1 gives that $R^1(p_2)_*\mathcal{A} = H^1\big(\mathbb{P}^1;\ \mathcal{A}|_{\{s\}}\big)$ for every $s \in S$ [12, II.5, Corollary 3]. Since S is affine, Theorem A of Serre gives the existence of $z \in H^0\big(S;\ R^1(p_2)_*\mathcal{A}\big)$ such that $z(0) = \epsilon$. The family of pairs $\big\{\big(\mathbb{E}|_{\{s\}}, z(s)\big)\big\}_{s\in S}$ gives a flat family of ropes X as fiber over 0 and smoothable general fiber. Following more details from [9] one can also obtain embedded deformations. □

References

[1] E. Arbarello and M. Cornalba, *Su una congettura di Petri*, Comm. Math. Helv. **56** (1981), no. 1, 1–38.

[2] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, Geometry of algebraic curves. I, Springer, Berlin, 1985.

[3] E. Ballico, *A remark on linear series of general k-gonal curves*, Boll. UMI (7), **3–A** (1989), no. 2, 195–197.

[4] E. Ballico, *Scrollar invariants of smooth projective curves*, J. Pure Appl. Algebra **166** (2003), no. 3, 239–246.

[5] D. Bayer and D. Eisenbud, *Ribbons and their canonical embeddings*, Trans. Amer. Math. Soc. **347** (1995), no. 3, 719–756.

[6] K. A. Chandler, *Geometry of dots and ropes*, Trans. Amer. Math. Soc. **347** (1995), no. 3, 767–784.

[7] M. Coppens and G. Martens, *Linear series on a general k-gonal curve*, Abh. Math. Sem. Univ. Hamburg **69** (1999), 347-371.

[8] D. Eisenbud and A. Van de Ven, *On the normal bundles of smooth rational space curves*, Math. Ann. **256** (1981), no. 4, 453–463.

[9] F. J. Gallego, M. González and B. P. Purnaprajna, *Deformation of finite morphisms and smoothing of ropes*, Compositio Math. **144** (2008), no. 3, 673–688.

[10] M. González, *Smoothing of ribbons over curves*, J. Reine Angew. Math. **591** (2006), 201–213.

[11] R. Hartshorne, *Cohomological dimension of algebraic varieties*, Ann. of Math. **88** (1968), no. 3, 402–450.

[12] D. Mumford, Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Oxford University Press, London 1970.

[13] G. Sacchiero, *Normal bundles of rational curves in projective space*, Ann. Univ. Ferrara Sez. VII (N. S.) **26** (1980), 33–40.

[14] F.-O. Schreyer, *Syzygies of canonical curves and special linear series*, Math. Ann. **275** (1986), no. 1, 105–137.

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