Vector bundles on a neighborhood of an exceptional curve and elementary transformations

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Abstract

Let W be the germ of a smooth complex surface around an exceptional curve and let E be a rank 2 vector bundle on W. We study the cohomological properties of a finite sequence $\{E_i\}_{1\leq i\leq t}$ of rank 2 vector bundles canonically associated to E. We calculate numerical invariants of E in terms of the splitting types of $E_i, 1 \leq j \leq t$. If S is a compact complex smooth surface and E is a rank two bundle on the blow-up of S at a point, we show that all values of $c_2(E)-c_2(p_*(E)^{\vee\vee})$ that are numerically possible are actually attained.

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1 Introduction

We consider exceptional curves in the following two cases. In the first case, let W be a smooth connected complex analytic surface which contains an exceptional divisor i.e. a smooth curve $D \simeq \mathbf{P}^1$ with $\mathcal{O}_D(-1)$ as normal bundle. Let U be a small tubular neighborhood of D in the Euclidean topology and let $p: U \to Z$ be the contraction of D. In this case Z is the germ of a smooth surface around the point $P := p(D)$. In the second case, let W be a smooth connected algebraic surface defined over an algebraically closed field K with arbitrary characteristic. We assume that W contains an exceptional curve D and denote by U the formal completion of W along D. Let $p: U \to Z$ be the contraction of D. In this case Z is a formal smooth 2-dimensional space supported at P. In what follows we use the notation defined above to represent either case. Let I be the ideal sheaf of D in U

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and consider a rank 2 vector bundle E over U. Consider the pair of integers (a,b) such that $E|_D \simeq \mathcal{O}_D(a) \otimes \mathcal{O}_D(b)$. We will refer to the pair (a,b) as the splitting type of E. Since Z is a smooth surface the bidual $p_*(E)^{\vee\vee}$ is locally free and hence free because Z is 2-dimensional. There is a natural inclusion $j\colon p_*(E) \to p_*(E)^{\vee\vee}$ such that $\mathrm{coker}(j)$ has finite length. Set $Q\colon= \mathrm{coker}(j)$. We show that the pair $(z,w)\colon= (h^0(Z,Q), h^0(Z,R^1p_*(E)))$ of numerical invariants of E is uniquely determined by a sequence of pairs of integers associated to E in [B] using elementary transformations. We review the construction of the associated sequence and prove the following results.

Theorem 0.1. *Let E be a rank 2 vector bundle on W with associated admissible sequence $\{(a_i,b_i)\}, 1 \le i \le t$. Then we have the equalities*

$$w := h^0(Z, R^1p_*(E)) = \sum_{1\le i\le t} max\{-b_i - 1, 0\}$$

and

$$z := h^0(Z,Q) = \sum_{1\le i\le t} a_i - a_t^2 - \sum_{1\le i\le t} max\{-b_i-1,0\}.$$

Every admissible sequence is associated to a rank 2 vector bundle on W (see [B] Th.0.2). For simplicity, we normalize our bundles to have splitting type $(j, -j+e)$, with $e=0$ or $e=-1$. We have the following existence theorem.

Theorem 0.2. *For every pair of integers (z,w) satisfying $j-1-e \le w \le j(j-1)/2 - j\,e$ and $1 \le z \le j(j+1)/2$ with $j \ge 0$ and $e \ge 0$ or -1, there exists a rank 2 vector bundle E on W with splitting type $(j,-j+e)$ having numerical invariants $h^0(Z,R^1p_*(E)) = w$ and $h^0(Z,Q) = z$.*

Remark 0.3. It follows from theorem 0.2 that the strata defined in [BG] for spaces of bundles on the blow-up of $\mathbf{C}^2$ are all non-empty. We give also the following characterization of the split bundle.

Proposition 0.4. *Let E be a rank 2 vector bundle on U with splitting type $(j,-j+e)$ with $j>0$ and $e=0$ or -1. The following conditions are equivalent:*

(i) $E \simeq \mathcal{O}_U(-jD) \oplus \mathcal{O}_U((j+e)D)$

(ii) $c_2(E) - c_2(p_*(E)^{\vee\vee}) = j(j+e)$

(iii) $h^0(Z, R^1p_*(E)) = j(j-1)/2$

(iv) E has associated sequence $\{(ai, b_i)\}, 1 \leq i \leq j-e$ *with* $b_i = -j-e+i-1$ *for every* i.

We now consider a compact complex smooth surface S, so that we can calculate second chern classes. If E is a rank 2 bundle defined on the blow-up of S at a point, then the difference of second Chern classes satisfies $j \leq c_2(E) - c_2(p_*(E)^{\vee\vee}) \leq j^2$ and is given by the sum $h^0(Z, R^1p_*(E)) + h^0(Z, Q)$ (see [FM]). Sharpness of these bounds was proven in [B] and in [G2] by different methods. We prove the following existence theorem.

Theorem 0.5. *Let S be the blow-up of a compact complex smooth surface S at a point. Let l denote the exceptional divisor and let j be a non-negative integer. Then for every integer k satisfying $j \leq k \leq j^2$ there exists a rank 2 vector bundle E over S with $E|l \simeq \mathcal{O}_l(j) \oplus \mathcal{O}_l(-j)$ satisfying $c_2(E) - c_2(p_*(E)^{\vee\vee}) = k$.*

Note 0.6: In [G1, Thm. 3.5] it is shown that the number of moduli for the space of rank-2 bundles on the blow up of $\mathbf{C}^2$ at the origin with splitting type j equals $2j-3$; and since such bundles are determined by their restriction to a formal neighborhood of the exceptional divisor it follows that we have the same number of moduli for bundles over the neighborhood U of an exceptional curve on a surface W. These results are proven in section 1, where we also review the construction of admissible sequences. On section 2 we consider briefly bundles of higher rank.

2 Rank 2 bundles

We briefly recall the construction of the associated sequences of pairs of bundles and splitting types given in the introduction of [B]. We first give the definitions of positive and negative elementary transformations. Let E be a rank 2 vector bundle on W with splitting type (a, b) with $a \geq b$. Fix a line bundle R on D and a surjection $r: E \to R$ induced by a surjection $\rho: E|_D \to R$. There exists such a surjection if and only if $deg(R) \geq b$. If $deg(R) = b < a$, then ρ is unique, up to a multiplicative constant. Set $E' := ker(r)$ and $R' := ker(\rho)$. If $deg(R) = b < a$ the sheaf E' is uniquely determined, up to isomorphism. Since D is a Cartier divisor, E' is a vector bundle on U. We will say that E' is the bundle obtained from E by making the negative elementary transformation induced by r. Note that R' is a line bundle on D with degree $deg(R') = a + b - deg(R)$. Since $deg(I/I^2) = 1$ it is easy to check that $deg(E'|_D) = a + b + 1$ and we have the exact sequence

$$0 \to \mathcal{O}_D(1 + deg(R)) \to E'|D \to R' \to 0. \tag{1}$$

Furthermore, using this exact sequence we obtain a surjection $t: E' \to R'$ such that $\ker(t) \simeq E(-D)$. In particular $\ker(t)|_D \simeq \mathcal{O}_D(a+1) \oplus \mathcal{O}_D(b+1)$. Thus, up to twisting by $\mathcal{O}_U(-D)$, the negative elementary transformation induced by r has an inverse operation and we will say that E is obtained from E ' making a positive elementary transformation supported by D. The following diagram, called the *display* of the elementary transformation, summarizes the construction (see [M]).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & R' & \longrightarrow & E|_D & \stackrel{\rho}{\longrightarrow} & R & \longrightarrow & 0 \\
 & & t\uparrow & & \uparrow & & || & & \\
0 & \longrightarrow & E' & \longrightarrow & E & \stackrel{r}{\longrightarrow} & R & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & E(-D) & = & E(-D) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Given two vector bundles E_1 and E_2 with splitting types (a_1, b_1) and (a_2, b_2) we say that E_1 is more balanced then E_2 if$a_1 - b_1 \leq a_2 - b_2$. Given a vector bundles E with splitting type (a, b) we say that E is balanced if either $a = b$ (case c_1 even) or else $a = b + 1$ (case c_1 odd). Performing negative elementary transformations we will take the bundle E into more balanced bundles. The sequence of elementary transformations finishes when we arrive at a balanced bundle. If $\deg(R) = b$, then $E'|_D$ fits in the exact sequence

$$0 \to \mathcal{O}_D(b+1) \to E'|D \to \mathcal{O}_D(a) \to 0. \tag{2}$$

If $b < a$ then E' is more balanced than E. If $b = a - 3$, then (2) does not uniquely determine $E'|_C$. If $b = a - 2$ and E' is not balanced, we reiterate the construction starting from E' taking R' to be the factor of $E'|D$ of lowest degree and we take the unique surjection (up to a multiplicative constant) $\rho': E' \to R'$. In a finite number, say, $t-1$, of steps, we send E into a bundle which, up to twisting by $\mathcal{O}_U(sD)$, where $s = (a+b+t-1)/2$ has trivial restriction to D. The process ends with a bundle isomorphic to $\mathcal{O}_U(sD)^{\oplus 2}$ (see [B], Remark 0.1).

We now construct the admissible sequence associated to E. Step one: set $E_1 := E$, $a_1 := a$ and $b_1 := b$. If $a_1 = b_1$, set $t = 1$ and stop. Otherwise $a_1 > b_1$. Step two: in the case $a_1 > b_1$ set $E_2 := E'$ and let (a_2, b_2) be the splitting type of E'. Note that $a_2 + b_2 = a_1 + b_1 + 1$ and $b_1 < b_2 = a_2 = a_1$. Hence $a_2 - b_2 < a_1 - b_1$ and E_2 is more balanced than E_1. If $a_2 = b_2$, set $t := 2$ and stop. If $a_2 > b_2$ reiterate the construction. Final step: in a finite number of steps (say $t-1$ steps) we arrive at a bundle E_t with splitting type (a_t, b_t) with $a_t = b_t$. Call $E_i, 2 \leq i \leq t$, the bundle obtained after $i-1$ steps and let (a_i, b_i) be the splitting type of E_i. The finite sequence of pairs $\{(a_i, b_i)\}, 1 \leq i \leq t$ obtained in this way has the following properties:

(i) $a_i \geq b_i \forall i > 0$,

(ii) $a_i + b_i = a_1 + b_1 + i - 1 \forall i > 1$,

(iii) $a_i \geq a_i + 1 \geq b_i + 1 > b_i, \forall i \geq 1$, and

(iv) $a_t = b_t$.

We call *admissible* any such finite sequence of pairs of integers. We will say that a sequence $\{(a_i, b_i)\}, 1 \leq i \leq t$ is the admissible sequence associated to the bundle E if this sequence is created by the algorithm just described. By [B] Th. 0.2, every admissible sequence is associated to a rank 2 vector bundle on W.

Examples: Let us first set some notation. To represent the admissible sequence $\{(a_i, b_i)\}, 1 \leq i \leq t$, we write

$$(a_1, b_1) \to (a_2, b_2) \to \cdots \to (a_t, b_t).$$

1. If the splitting type of E is $(b+2, b)$ then there is only one possibility for

the admissible sequence associated to E, namely

$$(b+2,b) \to (b+2,b+1) \to (b+2,b+2).$$

2. If the splitting type of E is $(b+4,b)$ then there are 3 different possibilities for admissible sequences associated to E (which in particular will give rise to different values of the numerical invariants (z,w)), these are:

i.$(b+4,b) \to (b+4,b+1) \to (b+4,b+2) \to (b+4,b+3) \to (b+4,b+4)$

ii.$(b+4,b) \to (b+4,b+1) \to (b+3,b+3)$

iii.$(b+4,b) \to (b+3,b+2) \to (b+3,b+3)$

We now calculate the numerical invariants of E in terms of admissible sequences. For every integer $n \geq 0$ let $D^{(n)}$ be the n-th infinitesimal neighborhood of D in U. Hence $D^{(n)}$ is the closed subscheme of U with I^{n+1} as ideal sheaf. In particular, $D^{(0)} = D$ and $D^{(n)}_{red} = D$ for every $n \geq 0$. For each integer $n \geq 0$ the following sequence is exact

$$0 \to I^n/I^{n+1} \to \mathcal{O}_U/I^{n+1} \to \mathcal{O}_U/I^n \to 0. \tag{3}$$

Suppose that E is a vector bundle normalized to have splitting type $(j,-j+e)$ where $j \geq 1$ and either $e=0$ or $e=-1$. We denote by m be the maximal ideal of $\mathcal{O}_{Z,P}$. Consider the inclusion $j : p_*(E) \to p_*(E)^{\vee\vee}$ and let $Q := coker(j)$, $z := h^0(Z,Q)$, and $w := h^0(Z, R^1p_*(E))$. Call $\mathcal{O}_D(x)$ the degree x line bundle on D. Twisting the exact sequence (3) by E and using the fact that I^n/I^{n+1} has degree n, we obtain the exact sequence

$$0 \to \mathcal{O}_D(j+n) \to \mathcal{O}_D(-j+e+n) \to E|_D(n) \to E|_D(n-1) \to 0. \tag{4}$$

Lemma 1.1. *The integers z and w satisfy the inequalities:*

$$1 \leq z \leq j(j+1)/2$$

and

$$j-1 \leq w \leq j(j-1)/2 - ej.$$

Proof. By the Theorem on Formal Functions we have the bounds for z and we have that

$$w = \sum_{n\geq 0} h^1(D, \mathcal{O}_D(-j+e+n)) = j(j-1)/2 - ej.$$

The upper bound for $w+z$ was stated in [FM] Remark 2.8, and proven for bundles with arbitrary rank in [Bu] Prop.2.8. Consequently we have an alternative proof of the upper bound for z. The lower bound for w will be proven in Remark 1.4. For the case of rank two and $e = 0$ [G2] shows that these bounds are sharp. Since Q is a quotient of $\mathcal{O}_{U,P}^{\oplus 2}$ the dimension of the fiber of Q at P is either 1 or 2. The sheaf Q is isomorphic to the structure sheaf of a subscheme of Z supported by P and with length z if and only if the dimension of this fiber is 1. We will check that this is always true (see Proposition 1.3). We first check the split case.

Lemma 1.2. *Suppose that $E \simeq \mathcal{O}_U(-jD) \oplus \mathcal{O}_U((j-e)D)$ then we have $z = j(j+1)/2, w = j(j-1)/2 - ej$ and Q is isomorphic to the structure sheaf of a subscheme of Z supported by P and with m^j as ideal sheaf.*

Proof. Since D is an exceptional divisor, we have $p_*(\mathcal{O}_U((j-e)D)) = p_*(\mathcal{O}_U) = \mathcal{O}_Z$ for every $j = e$ and $p_*(\mathcal{O}_U(-jD)) \simeq m^j$ if $j > 0$.

Proposition 1.3. *Let E be a rank 2 vector bundle on W having splitting type $(j, -j+e)$ with $j > 0$. Then Q is isomorphic to the structure sheaf of a length z subscheme Q of Z with $Q_{red} = P$ and $Q \subset P^{(j-1)}$.*

Proof of 1.3. The first assertion is well-known and follows from the proof of Lemma 1.2. Since Q is a quotient of $\mathcal{O}Z^{\oplus 2}$, in order to prove the second assertion it is sufficient to check that its fiber at P is a 1-dimensional vector space. Since E has splitting type $(j, -j+e)$, we have an extension

$$0 \to \mathcal{O}_U((-j+e)D) \to E \to \mathcal{O}_U(jD) \to 0. \tag{5}$$

([BG] Lemma 1.2, or in [G1] Thm. 2.1 in the case $e = 0$). Call $\mathbf{e}$ the extension (5) giving E. For each $t \in K - 0$ consider the extension of $\mathcal{O}_U(jD)$ by $\mathcal{O}_U((-j+e)D)$ given by extension class $t\,\mathbf{e}$, this extension has as middle term a vector bundle isomorphic to E. Using the extension $\mathbf{e}$ for $t = 0$, we construct a family $\{\lambda \mathbf{e}\}_{\lambda\in K}$ of extensions. We call E_λ the corresponding middle term and Q_λ the corresponding sheaf. Since $E_\lambda \simeq E$ for $\lambda \neq 0$, we have $Q_\lambda = Q$ for $\lambda \neq 0$, and because $E_0 \simeq \mathcal{O}_W(jD) \oplus \mathcal{O}_W((-j+e)D)$, we have that $Q_0 = P^{(j-1)}$, and the result follows by semi–continuity of the fiber dimension at P.

Proof of 0.1. Given the admissible sequence of splitting types $\{(a_i, b_i)\}_{1\leq i\leq t}$ associated to E we want to show that

$$w := h^0(Z, R^1p_*(E)) = \sum_{1\leq i\leq t} max\{-b_i - 1, 0\}$$

and

$$z := h^0(Z, Q) = \sum_{1\leq i\leq t} a_i - a_t^2 - \sum_{1\leq i\leq t} max\{-b_i - 1, 0\}.$$

We use induction on t, the case $t = 1$ arising if and only if $a_1 = b_1$, equivalently, when $E \simeq \mathcal{O}_W(-a_1D)^{\oplus 2}$ (this follows immediately from the definition of admissible sequence). Since $R^1p_*(\mathcal{O}_W(xD)) = 0, \forall x \leq 1$ and $R^1p_*(\mathcal{O}_W(yD)) = y(y-1)/2, \forall y > 0$, we have the equality for w in the split case. Assume $t \geq 2$. By the definition of the sequence $\{E_i\}, 1 \leq i \leq t$ associated to E we have that $E_1 = E$ and there is an exact sequence

$$0 \to E_2 \to E_1 \to \mathcal{O}_D(-b_1D) \to 0. \tag{6}$$

First assume $b_1 < 0$, in which case we have that $h^0(Z, p_*(\mathcal{O}_D(-b_1D))) = 0$ and $h^0(Z, R^1p_*(\mathcal{O}_W(-b_1D))) = -b_1 - 1$. Hence $w := h^0(Z, R^1p_*(E)) = h^0(Z, R^1p_*(E_2)) - b_1 + 1$ and since E_2 has $\{(a_{i+1}, b_{i+1})\}, 1 \leq i < t$ as admissible sequence, the claim follows.

Now assume $b_1 \geq 0$, from the exact sequence (6) it follows that

$$h^0(Z, R^1p_*(E)) \leq h^0(Z, R^1p_*(E_2)).$$

Since $b_i > b_1$ for every $i > 1$, we have $h^0(Z, R^1p_*(E_2)) = 0$. Hence, by the inductive assumption on the length of the admissible sequence, it follows that $h^0(Z, R^1p_*(E)) = 0$, proving the first assertion. The value of $z := h^0(Z, Q)$ comes from the equalities

$$c_2(E) - c_2(p_*(E)^{\vee\vee}) = \sum_{1\leq i<t} a_i - a_t^2$$

and

$$c_2(E) - c_2(p_*(E)^{\vee\vee}) = h^0(Z, Q) + h^0(Z, R^1p_*(E))$$

proved in [B, Th. 0.3] and in [FM] respectively. Here, of course, we assume that E is extended to a compactification, however these integers do not depend upon the choices of compactification and of extension of E.

Proof of 0.2. By [B] Th. 0.2 every admissible sequence (a_i, b_i) is associated to a rank two bundle E on W, moreover, the intermediate steps of the construction of E give bundles E_i with splitting types (a_i, b_i) for each i. Now use Th. 0.1 to calculate z and w.

Remark 1.4. If we assume that E has splitting type $(j, -j+e)$ with $j \geq 1+e$, then because $b_1 = -j + e$, we obtain $w \geq j - 1 - e$.

Proof of 0.4. By [B] Th. 0.5 we know that (i) and (ii) are equivalent. By Lemma 1.2 (i) implies (iv). Since $b_1 = -j + e$, and $b_i > b_i - 1$ holds $\forall i > 1$, and since $a_1 = j$, and $a_i + b_i = e + i - 1$ holds $\forall i > 1$; it follows from Theorem 0.1 that (iv) implies (ii).

Proof of 0.5. Given bundles G on S and F on W with $c_1(G) = 0 = c_1(F)$ there exists a bundle E on S satisfying $E|_{S-l} = p_*E|_{S-\{p\}}$ and $E|_W = F$ (see [G3] Cor. 3.4).It then follows that $c_2(E) - c_2(p_*(E)^{\vee\vee}) = R^1p_*(F) + l(Q)$ and the result follows from Th. 0.2.

3 Bundles of higher rank

In this section we consider vector bundles with rank $r \geq 3$. Fix a rank r vector bundle E on U. We use the notation of [B] for the admissible sequence $\{E_i\}, 1 \leq i \leq t$ of vector bundles associated to E. In particular we denote by $(a(i,1), ..., a(i,r))$ the splitting type of E_i where $a(i,1) \geq \cdots \geq a(i,r)$. We make the strong assumption that $a(i, r-1) \geq -1$ for every i and compute $h^0(Z, R^1p_*(E))$.

Proposition 2.1. *Let E be a rank r vector bundle on W whose associated sequence of vector bundles $\{E_i\}$ has splitting type $(a(i,1), ..., a(i,r))$ with*

$a(i, r-1) \geq -1$, *for all* i $1 \leq i \leq t$. *Then we have*

$$h^0(Z, R^1p_*(E)) = \sum_{1 \leq i \leq t} min\{-a(r,i) - 1, 0\}.$$

Proof. We first observe that the proof of the corresponding inequality for rank 2 bundles works verbatim (both cases $t = 1$ and $t > 1$), because, for each i with $1 \leq i \leq t$ at most one of the integers $a(i,j)$ is not at least -1 and $h^1(\mathbf{P}^1, L) = 0$ for every line bundle L on $\mathbf{P}^1$ with $\deg(L) \geq -1$. In the case $r \geq 3$, the sequence of elementary transformations made to balance the bundle is not, a priori, uniquely determined, and hence the sequence of associated bundles is not uniquely determined by E. The condition $a(1, r-1) \geq -1$ implies that there is an associated sequence in which we make always an elementary transformation with respect to $\mathcal{O}_D(a(r,i))$ to pass from E_i to E_{i+1} for some $a(r,i) \leq -1$ (which gives that $h^0(Z, R^1p_*(Ei)) = h^0(Z, R^1p_*(Ei+1)) - a(r,i) + 1$). We continue to perform elementary transformations until we arrive at an integer $m \leq t$ such that $a(m,j) \geq -1$ for every i. It is then quite easy to check that $h^0(Z, R^1p_*(Em)) = 0$ and the result follows.

In the general case the same method gives the following partial result.

Proposition 2.2. *Let* E *be a rank* r *vector bundle on* W *whose associated sequence* $\{E_i\}, 1 \leq i \leq t$ *of vector bundles has splitting type* $(a(i,1), ..., a(i,r))$ *with* $1 \leq i \leq t$. *Then we have*

$$h^0(Z, R^1p_*(E)) = \sum_{\substack{1 \leq i \leq t \\ 1 \leq j \leq r}} min\{-a(j,i) - 1, 0\}.$$

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