Rank Two Bundles on the Blow-up of $\mathbf{C}^2$

Elizabeth Gasparim
Departamento de Matematica, Universidade Federal de Pernambuco
Cidade Universitaria, Recife, PE, BRASIL, 50760-380

Abstract

In this paper we study holomorphic rank two vector bundles on the blow up of $\mathbf{C}^2$ with vanishing Chern class. The restriction of such a bundle over the exceptional divisor splits as $\mathcal{O}(j) \oplus \mathcal{O}(-j)$ for some integer j. We denote by $\mathcal{M}_j$ the moduli space of holomorphic bundles on the blow up of $\mathbf{C}^2$ whose restriction to the exceptional divisor is $\mathcal{O}(j) \oplus \mathcal{O}(-j)$. We prove that $\mathcal{M}_j$ is generically a complex projective space of dimension $2j-3$.

1 Introduction

Holomorphic vector bundles over complex surfaces have been extensively studied by several different methods. See for example the books of Kobayashi [9], Okonek, Schneider, Spindler [10] and Donaldson, Kronheimer [2]. A fundamental result on the classification of rational surfaces is: "Every rational surface is obtained by blowing up points on either $\mathbf{P}^2$ or on a rational ruled surface"(see Griffiths and Harris [6]). This result suggests that the understanding of vector bundles on rational surfaces depends on the analysis of the behavior of vector bundles under blow-ups.

Some works on holomorphic bundles on blow-ups are the papers by Freedman and Morgan [3][4], Brussee [1], and Qin [11]. Roughly speaking we may see the "difference" between moduli spaces of bundles on a rational surface and moduli spaces of bundles on one of its minimal models by studying bundles on the blow up of $\mathbf{C}^2$. In this work we concentrate on the study of bundles on blow-ups in the local sense, that is in a neighborhood of the exceptional divisor. Our approach is quite concrete, as we give bundles explictly by their transition matrices and present the moduli spaces as quotients of $\mathbf{C}^n$ by an equivalence relation.

1.1 Statement of Results

We use the following notations:
$\widetilde{\mathbf{C}}^2$ = the blow up of $\mathbf{C}^2$ at the origin
ℓ = the exceptional divisor
E_ℓ = restriction of the bundle E to ℓ.

Since we consider rank two bundles with zero first Chern class we must have $E_\ell = \mathcal{O}(j) \oplus \mathcal{O}(-j)$ for some integer j, by Grothendieck's theorem.

Definition: $\mathcal{M}_j$ is defined as the moduli space of bundles on $\widetilde{\mathbf{C}}^2$ which restrict to $\mathcal{O}(j) \oplus \mathcal{O}(-j)$ over the exceptional divisor.

For a bundle in $\mathcal{M}_j$ we give a canonical form of transition matrix, from which we have two immediate corollaries.

Corollary 2.3:$\mathcal{M}_0$ *consists of a single point.*

Corollary 2.5: $\mathcal{M}_1$ *consists of a single point.*

Our main results are:

Theorem 3.4: *The moduli space $\mathcal{M}_2$ is homeomorphic to the union $\mathbf{P}^1 \cup \{p, q\}$, of a complex projective plane $\mathbf{P}^1$ and two points, with a basis of open sets given by*

$$\mathcal{U} \cup \{p, U : U \in \mathcal{U} - \phi\} \cup \{p, q, U : U \in \mathcal{U} - \phi\}$$

where $\mathcal{U}$ is a basis for the standard topology of $\mathbf{P}^1$.

Theorem 3.5: *The generic set of the moduli space $\mathcal{M}_j$ is a complex projective space of dimension $2j - 3$ (minus a closed subvariety of complex codimension bigger than or equal to two).*

Remark: The moduli space $\mathcal{M}_j$ also contains complex projective spaces of every dimension smaller than $2j - 3$, each one deleted of some closed subvariety.

Remark: $\mathcal{M}_j$ is not Hausdorff. For example, the direct sum bundle given by $\begin{pmatrix} z^j & 0 \\ 0 & z^{-j} \end{pmatrix}$ is "arbitrarily close" to any other bundle.

2 The canonical form of transition matrix

We write $\widetilde{\mathbf{C}}^2 = U \cup V$, where $U = \mathbf{C}^2 = \{(z,u)\}$, $V = \mathbf{C}^2 = \{(\xi, v)\}$, $U \cap V = (\mathbf{C} - \{0\}) \times \mathbf{C}$ with the change of coordinates $(\xi, v) = (z^{-1}, zu)$. Naturally, we first look at line bundles. It turns out that holomorphic line bundles on $\widetilde{\mathbf{C}}^2$ are classified by their Chern classes. This can be easily seen using the exponential sheaf sequence and the fact that $H^1(\mathcal{O}(-1), \mathcal{O}) = 0$. For rank two bundles with coordinate charts U and V as above we give a canonical form of transition matrix.

Theorem 2.1 : *Let E be a holomorphic bundle on $\widetilde{\mathbf{C}}^2$ satisfying $E_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j)$. Then E has a transition matrix of the form*

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$$

from U to V, where

$$p = \sum_{i=1}^{2j-2} \sum_{l=i-j+1}^{j-1} p_{il} z^l u^i.$$

In particular the bundle E is algebraic.

Proof: Because $E_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j)$, a transition matrix for E from U to V takes the form

$$T = \begin{pmatrix} z^j + ua & uc \\ ud & z^{-j} + ub \end{pmatrix}$$

where a, b, c, and d are holomorphic functions in $U \cap V$. In order to obtain the desired form of the matrix, we change coordinates three times as follows. Step I. Shows E is an extension using the change of coordinates

$$\begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \begin{pmatrix} z^j + ua & uc \\ ud & z^{-j} + ub \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}.$$

Step II. Shows $0 \to \mathcal{O}^!(-j) \to E \to \mathcal{O}^!(j) \to 0$ is exact using the change of coordinates

$$\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \begin{pmatrix} z^j + ua & uc \\ 0 & z^{-j} + ub \end{pmatrix} \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}.$$

Step III. Finds the formula for the polynomial p using the coordinate change

$$\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^j & uc \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}.$$

In each of these Steps the ξ's and the η's are holomorphic functions in U and V respectively. All three steps use the same technique, so it is enough to see the detailed proof of Step 1, which follows.
Proof of Step I: As indicated above, in this Step we perform the coordinate changes

$$\begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \begin{pmatrix} z^j + ua & uc \\ ud & z^{-j} + ub \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}.$$

After this multiplication the entry $e(2,1)$ of the resulting matrix is

$$e(2,1) = \eta(z^j + ua) + ud + [\eta uc + (z^{-j} + ub)]\,\xi.$$

The term independent of u in $e(2,1)$ is $\eta_0(z^{-1})\,z^j + \xi_0(z)\,z^{-j}$. Choosing $\eta_0(z^{-1}) = -z^{-j}$ and $\xi_0(z) = z^j$ we cancel this term and $e(2,1)$ becomes a multiple of u.

Inductively, assume that the values of $\eta_0, \eta_1, ..., \eta_{n-1}$ and $\xi_0, \xi_1, ..., \xi_{n-1}$ have been chosen so that they cancel the coefficients of $u^0, u^1, ...u^{n-1}$ in the expression for $e(2,1)$. The coefficient of u^n in the expression for $e(2,1)$ is

$$\eta_n(z^{-1})\,z^{j+n} + \sum_{m+i=n} \xi_m(z)\,b_i(z, z^{-1}) + \Phi^n(z, z^{-1}),$$

where $\Phi^n(z, z^{-1})$ is a holomorphic function on z and z^{-1}. We separate Φ^n into $\Phi^n = \Phi^n_{\geq 0} + \Phi^n_{<0}$ where $\Phi^n_{\geq 0}$ is the part of Φ^n containing the non-negative powers of z and $\Phi^n_{<0}$ is the part of Φ^n containing the negative powers of z. We then choose the values of η_n and ξ_n as $\eta_n = z^{-n-j}\Phi^n_{<0}$ and $\xi_n = z^j\,\Phi^n_{\geq 0}$. These choices cancel the coefficient of u^n in the expression for $e(2,1)$. ∎

Remark: In [8] J. Hurtubise finds a similar form of transition matrix for bundles on a product $\mathbf{P}^1 \times U$ where U in an open set in $\mathbf{C}$, using Grothendieck's Theorem on Formal Functions. A similar approach does not work in our case, as it depends on the vanishing of a cohomology group, which does not happen on the blow up. However, it is possible to prove Hurtubise's Theorem for bundles on $\mathbf{P}^1 \times U$ using our method of coordinate changes.

2.1 The moduli spaces $\mathcal{M}_0$ and $\mathcal{M}_1$

It turns out that for each of the cases $j=0$ and $j=1$ there is only one possible bundle up to isomorphism.

Corollary 2.2 *: A holomorphic rank two vector bundle E on $\widetilde{\mathbf{C}}^2$ which is trivial when restricted to the exceptional divisor is trivial on $\widetilde{\mathbf{C}}^2$.*

Proof: Let $j=0$ in Theorem 3. Then $p=\sum_{i=1}^{-2}\sum_{l=i}^{-1} p_{il}z^l u^i = 0$ and the transition matrix for E is the identity. ∎

We may restate this as:

Corollary 2.3 *:$\mathcal{M}_0$ consists of a single point.*

Corollary 2.4 *: A holomorphic rank two vector bundle E on $\widetilde{\mathbf{C}}^2$ which has the restriction $E_\ell \simeq \mathcal{O}(1)\oplus\mathcal{O}(-1)$ splits into a sum of line bundles.*

Proof: Let $j=1$ in Theorem 2. Then $p=\sum_{i=1}^{0}\sum_{l=i}^{0} p_{il}z^l u^i = 0$ and the transition matrix for E is $\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}$. ∎

We restate the previous Corollary as

Corollary 2.5 *: $\mathcal{M}_1$ consists of a single point.*

3 The moduli spaces $\mathcal{M}_j$

Next we investigate when two distinct transition matrices from Theorem 3 give holomorphically equivalent vector bundles.

Lemma 3.1 *: If $p' = \lambda p$ for some $\lambda \in \mathbf{C}-\{0\}$, then the matrices*

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j}\end{pmatrix} \text{ and } \begin{pmatrix} z^j & p' \\ 0 & z^{-j}\end{pmatrix}$$

give holomorphically equivalent vector bundles.

Proof: Just write down the isomorphism as

$$\begin{pmatrix} z^j & p' \\ 0 & z^{-j}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda\end{pmatrix}\begin{pmatrix} z^j & p \\ 0 & z^{-j}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \lambda\end{pmatrix}.$$ ∎

To see that the value $\lambda = 0$ must indeed be excluded from the above proposition we have the

Example: The holomorphic bundles E and F, given by transition matrices

$$\begin{pmatrix} z^2 & zu^2 \\ 0 & z^{-2} \end{pmatrix} \ \textit{and} \ \begin{pmatrix} z^2 & 0 \\ 0 & z^{-2} \end{pmatrix}$$

respectively are not holomorphically equivalent. In other words, the bundle E does not split.

Recall that if Y is a closed subvariety of a variety X, defined by the sheaf of ideals $\mathcal{I}$, then the n-th formal neighborhood of Y in X is the quotient $\mathcal{O}_X/\mathcal{I}^{n+1}$. To simplify the notation we will write our polynomial p as $p = p_1 + p_2 + \cdots + p_{2j-2}$ where p_i is the term in u^i.

Proposition 3.2 : *On the first formal neighborhood, two bundles $E^{(1)}$ and $E^{(1)\prime}$ with transition matrices*

$$\begin{pmatrix} z^j & p_1 \\ 0 & z^{-j} \end{pmatrix} \ \textit{and} \ \begin{pmatrix} z^j & p_1' \\ 0 & z^{-j} \end{pmatrix}$$

respectively are isomorphic if and only if $p_1' = \lambda p_1$ for some $\lambda \in \mathbf{C} - \{0\}$.

Proof: The if part follows from Lemma 5. Now suppose $E^{(1)}$ and $E^{(1)\prime}$ are isomorphic. According to our notation we have $p_1 = \sum_{l=2-j}^{j-1} p_{1l} z^l u$ and $p_1' = \sum_{l=2-j}^{j-1} p_{1l}' z^l u$. We will write the isomorphism in the form

$$\begin{pmatrix} z^j & p_1' \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z^j & p_1 \\ 0 & z^{-j} \end{pmatrix}.$$

where a, b, c, and d are holomorphic in U, and α, β, γ, and δ are holomorphic in V. On the first formal neighborhood, this yelds the following set of equations

$$\left\{ \begin{array}{rcll} (a_0(z) + a_1(z)u)\, z^j + p_1' c_0(z) & = & (\alpha_0(z^{-1}) + \alpha_1(z^{-1})zu)\, z^j & (1) \\ z^{-j}\,(c_0(z) + c_1(z)u) & = & (\gamma_0(z^{-1}) + \gamma_1(z^{-1})zu)\, z^j & (2) \\ (b_0(z) + b_1(z)u)\, z^j + p_1' d_0(z) & = & \alpha_0(z^{-1})p_1 + (\beta_0(z^{-1}) + \beta_1((z^{-1})zu)\, z^{-j} & (3) \\ z^{-j}\,(d_0(z) + d_1(z)u) & = & \gamma_0(z^{-1})p_1 + (\delta_0(z^{-1}) + \delta_1(z^{-1})zu)\, z^{-j}. & (4) \end{array} \right.$$

Recalling that p_1 and p_1' are multiples of u and equating terms that are independent of u in (1) and (4) gives $a_0(z) = \alpha_0(z^{-1})$ and $d_0(z) = \delta_0(z^{-1})$ respectively. Therefore a_0, α_0, d_0, and δ_0 are constants and $a_0 = \alpha_0$ and $d_0 = \delta_0$. Next we equate terms in u in equation (3), obtaining

$$b_1(z)u\,z^j + p_1' d_0 = \alpha_0 p_1 + \beta_1(z^{-1})u\,z^{-j},$$

which forces b_1 and β_1 to vanish. Equation (3) now becomes $p_1' d_0 = \alpha_0 p_1$ and we observe that p_1 and p_1' differ by a constant.

It remains to show that d_0 and α_0 are nonzero. Taking terms that are independent of u in equation (3) we have $b_0(z)z^j = \beta_0(z^{-j})z^{-j}$, which implies $b_0(z) = \beta_0(z^{-1}) = 0$. It follows that over the exceptional divisor our coordinate change has determinant $a_0 d_0$, hence $\alpha_0\delta_0 = a_0 d_0 \neq 0$ ∎

3.1 The moduli space $\mathcal{M}_2$

In the particular case when $j = 2$ our polinomial is $p = (p_{10} + p_{11}z)u + p_{21}zu^2$. We want to define a topology in $\mathcal{M}_2$ which is in some sense natural. For this we define a function $\Phi : \mathcal{M}_2 \to \mathbf{C}^3/\sim$ by $\begin{pmatrix} z^2 & p \\ 0 & z^{-2} \end{pmatrix} \to (p_{10}, p_{11}, p_{21})$. To make Φ a well defined function we need the appropriate equivalence relation $\sim$ which is given by:

i) $\{(p_{10}, p_{11}, p_{21}) \sim (\lambda p_{10}, \lambda p_{11}, p_{21}')\}$ *if* $(p_{10}, p_{11}) \neq (0,0)$, $\lambda \neq 0$

ii) $\{(0, 0, p_{12}) \sim (0, 0, \lambda p_{12})\}$, $\lambda \neq 0$.

Proposition 3.3 : *The map* $\Phi : \mathcal{M}_2 \to \mathbf{C}^3/\sim$ *is a bijection.*

Proof: We show that Φ is well defined. It is then easy to see that it is a bijection. Suppose we have isomorphic bundles with correponding polynomials $p = (p_{10} + p_{11}z)u + p_{21}zu^2$ and $p' = (p_{10}' + p_{11}'z)u + p_{21}'zu^2$. Based on Proposition 3.2, we know that p_1' and p_1 differ by a constant $d_0 p_1' = a_o p_1$. Without loss of generality we may assume $a_0 = d_0 = 1$ and consider $p \sim \lambda p$, for $\lambda \neq 0$. We write the isomorphism in the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z^2 & p' \\ 0 & z^{-2} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z^{-2} & -p \\ 0 & z^2 \end{pmatrix}$$

$$= \begin{pmatrix} a + z^{-2}p'c & z^4 b + z^2(p'd - ap) - pp'c \\ z^{-4}c & d - z^{-2}pc \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is holomorphic in z, u and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is holomorphic in z^{-1}, zu. We need to analyze what contraints this puts on p_{21} and p'_{21}. To begin with it follows that $c = \sum_{i-k\leq 4} c_{ik} z^i u^k$ with i, k both ≥ 0. Now looking at the (1,1) term we have that $\alpha = a + z^{-2}p'c$ must be holomorphic in z^{-1}, zu. This means that the coefficients of a must be chosen to cancel out in the expression for α all elements having $z^i u^j$ with $i > j$. Writing a in power series $a(z,u) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ji} z^i u^j$ and plugging into the equation for α we obtain the following constraints for a

$$\begin{pmatrix} a_{12} \\ a_{13} \end{pmatrix} = -\begin{pmatrix} p_{11} & p_{10} \\ 0 & p_{11} \end{pmatrix}\begin{pmatrix} c_{03} \\ c_{04} \end{pmatrix}$$

$$\begin{pmatrix} a_{23} \\ a_{24} \end{pmatrix} = -\begin{pmatrix} p_{11} & p_{10} \\ 0 & p_{11} \end{pmatrix}\begin{pmatrix} c_{14} \\ c_{15} \end{pmatrix} - \begin{pmatrix} p'_{21} \\ 0 \end{pmatrix}\begin{pmatrix} c_{04} \end{pmatrix}$$

and other equations for the coefficients of u^3, u^4 ... .

Using term (2,2) we get analogous equations for d, namely, if $d(z,u) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d_{ji} z^i u^j$, then

$$\begin{pmatrix} d_{12} \\ d_{13} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{10} \\ 0 & p_{11} \end{pmatrix}\begin{pmatrix} c_{03} \\ c_{04} \end{pmatrix}$$

$$\begin{pmatrix} d_{23} \\ d_{24} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{10} \\ 0 & p_{11} \end{pmatrix}\begin{pmatrix} c_{14} \\ c_{15} \end{pmatrix} + \begin{pmatrix} p_{21} \\ 0 \end{pmatrix}\begin{pmatrix} c_{04} \end{pmatrix}$$

and other equations for the coefficients of u^3, u^4 ... .

Looking at the (1,2) term we have that $\beta = z^4 b + z^2(p'd - ap) - pp'c$ is holomorphic in z^{-1}, zu. This means β has only terms $z^i u^k$ for $i \leq k$. We can ignore terms involving z^i for $i \geq 4$ since $z^4 b$ is available to remove them. So, to eliminate terms with $z^i u^k$ for $i > k$ we only need to consider the equation in terms up to u^2. This imposes only the condition that the terms in u and u^2 in the expression $z^2(p'd - ap) - pp'c$ be holomorphic in z^{-1}, zu. But the u-coefficient vanishes, so we only need to impose the condition that the coefficient of z^3u^2 be equal to zero, namely

$$(p'_{21} - p_{21}) - p_{10}^2 c_{03} - 2p_{10}p_{11}c_{02} - p_{11}^2 c_{01} + p_{10}(d_{11} - a_{11}) + p_{11}(d_{10} - a_{10}) = 0.$$

If $p_{10} + p_{11}z \neq 0$ the equation can be solved for any values of p_{21} and p'_{21} by choosing appropriate values of a_{10} and a_{11}, and we get relation i). If $p_{10} + p_{11}z = 0$ the equation implies $p_{21} = p'_{21}$, and we get relation ii). ∎

We now give $\mathcal{M}_2$ the topology induced from the correpondence Φ.

Theorem 3.4 *: The moduli space $\mathcal{M}_2$ is homeomorphic to the union $\mathbf{P}^1 \cup \{p, q\}$, of a complex projective plane $\mathbf{P}^1$ and two points, with a basis of open sets given by*

$$\mathcal{U} \cup \{p, U : U \in \mathcal{U} - \phi\} \cup \{p, q, U : U \in \mathcal{U} - \phi\}$$

where $\mathcal{U}$ is a basis for the standard topology of $\mathbf{P}^1$.

Proof: Take the topology on $\mathcal{M}_2$ induced by Φ. ■

Note that the moduli space $\mathcal{M}_2$ is not Hausdorff. Intuitively we can say that the direct sum bundle $\mathcal{O}(j) \oplus \mathcal{O}(-j)$ is "arbitrarily close" to any bundle on $\mathcal{M}_2$. The same statement holds if we replace the direct sum bundle by a bundle given by transition matrix $\begin{pmatrix} z^2 & \lambda z u^2 \\ 0 & z^{-2} \end{pmatrix}$. A similar detailed description of the moduli space $\mathcal{M}_3$ appears in [5].

3.2 The moduli space $\mathcal{M}_j$

Theorem 3.5 *The generic set of the moduli space $\mathcal{M}_j$ is a complex projective space of dimension $2j-3$ minus a closed subvariety of complex codimension bigger than or equal to two.*

Idea of the Proof: We define a function $\mathcal{M} \to \mathbf{C}^N / \sim$ by the rule $\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \to (p_{-j+2,1}; p_{-j+3,1}; ...; p_{j-1,2j-2})$, where the right side are just the coefficients of p. Generically, the restriction of our polinomial to the first formal neighborhood is nonzero. By tedious calculations similar to the ones in Proposition 3.3 one shows that in this case the higher formal neighborhoods are neglectable (except in a subset of codimension at least two). In the first formal neighborhood p_1 has $2j-2$ coefficients and from Proposition 3.2 we have the relation $p_1 \sim \lambda p_1$ which gives us the projective space of dimension $2-3$. ■

Remark: The moduli space $\mathcal{M}_j$ also contains complex projective spaces of every dimension smaller than $2j-3$, minus some closed subvariety. Projective spaces of dimension $2j-3-i$ appear from bundles which corresponding polynomial vanish up to the i-th formal neighborhood.

Remark: If we give $\mathcal{M}_j$ the topology induced from $\mathbf{C}^N$, then $\mathcal{M}_j$ is not Hausdorff. For example, the bundle $\mathcal{O}(j) \oplus \mathcal{O}(-j)$ is "arbitrarily close" to any other bundle in $\mathcal{M}_j$.

References

[1] Brussee, R. *Stable bundles on blown up surfaces. Math. Z. 205,551-565(1990)*

[2] Donaldson, S. K. and Kronhheimer, P. B. *The Geometry of Four Manifolds. Oxford University Press(1990)*

[3] Friedman, R. and Morgan, J. *On the diffeomorphism types of certain algebraic surfaces. I. J.Differ. Geom. 27,297-369(1988)*

[4] Friedman, R. and Morgan, J. *On the diffeomorphism types of certain algebraic surfaces II. J.Differ. Geom. 27,371-398(1988)*

[5] Gasparim, E. *Ph.D. thesis. University of New Mexico (1995)*

[6] Griffiths, P. and Harris, J. *Principles of Algebraic Geometry. John Wiley and Sons, Inc.(1978)*

[7] Hartshorne, R. *Algebraic Geometry. Graduate Texts in Mathematics 56, Springer Verlag (1977)*

[8] Hurtubise, J. *Instantons and Jumping Lines. Commun. Math. Phys. 105,107-122(1986)*

[9] Kobayashi, S. *Differential Geometry of Complex Vector Bundles. Princeton University Press(1978)*

[10] Okonek, C., Schneider, M., Spindler, H. *Vector Bundles on Complex Projective Spaces. Birkhäuser(1980)*

[11] Qin, Z. *Stable Rank-2 Sheaves on Blown-up Surfaces. unpublished*