

VECTOR BUNDLES ON A THREE DIMENSIONAL  
NEIGHBORHOOD OF A RULED SURFACE

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**Abstract**

Let  $S$  be a ruled surface inside a smooth threefold  $W$  and let  $E$  be a vector bundle on a formal neighborhood of  $S$ . We find minimal conditions under which the local moduli space of  $E$  is finite dimensional and smooth. Moreover, we show that  $E$  is a flat limit of a flat family of vector bundles whose general element we describe explicitly.

## 1 Introduction

Consider the general question: how do moduli spaces of vector bundles change under birational transformations of the base? In this paper, we take the first steps of a program to study this question for threefolds. In dimension three, flops give essential examples of birational transformations.

We first recall the definition of the basic flop. Let  $X$  be the cone over the ordinary double point defined by the equation  $xy - zw = 0$  on  $\mathbb{C}^4$ . The basic flop is described by the diagram:

$$\begin{array}{ccc}
 & \tilde{X} & \\
 f_1 \swarrow & & \searrow f_2 \\
 X_1 & & X_2 \\
 \pi_1 \searrow & & \swarrow \pi_2 \\
 & X &
 \end{array}$$

where  $\tilde{X} := \tilde{X}_{x,y,z,w}$  is the blow up of  $X$  at the vertex  $x = y = z = w = 0$ ,  $X_1 := \tilde{X}_{x,z}$  is the blow up of  $X$  along  $x = z = 0$  and  $X_2 := \tilde{X}_{y,w}$  is the blow up of  $X$  along  $y = w = 0$ . The *basic flop* is the transformation from  $X_1$  to  $X_2$ . The spaces appearing in this diagram are not compact, but they do contain neighborhoods of compact curves. We wish to find what vector bundles fit over this diagram, together with their local deformations. Note that on the given diagram, the spaces  $X_1$  and  $X_2$  are both abstractly isomorphic to

$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ , although the maps  $\pi_1$  and  $\pi_2$  are distinct; whereas  $\widetilde{X}$  is isomorphic to  $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$ ,

We generalize the situation of  $\widetilde{X}$  slightly by considering a ruled surface  $S$  with negative normal bundle inside a smooth threefold. We then study bundles  $E$  on a formal neighborhood  $\widehat{S}$  of  $S$  and their local moduli spaces (cf. definition ??). In the case of a Hirzebruch surface  $S$ , we require that  $E|_S$  be simple. When  $S$  is ruled over a curve  $C$  of genus greater than 1, we assume that  $E|_S$  is  $R$ -stable with respect to a good polarization  $R$  of  $S$  (cf. definition ??).

These conditions are minimal in the following sense. The local moduli space of a simple bundle is unobstructed, and therefore smooth (cf. remark ??). Hence to have smoothness it would be desirable to impose the condition that  $E$  be simple. However, as an easy argument in section 3 shows, there are no simple bundles on  $\widehat{S}$ . The alternative is to impose a condition on the restriction of  $E$  to an infinitesimal neighborhood of  $S$ . We choose the zero-th formal neighborhood. We have the following results.

**Theorem A** *Let  $S$  be a Hirzebruch surface with negative normal bundle inside a smooth threefold. Let  $E = \{E_n\}$  be a vector bundle on  $\widehat{S}$  such that  $E|_S$  is simple. Then the local moduli space of  $E$  is finite dimensional and smooth. Moreover,  $E$  is a flat limit of a flat family of vector bundles on  $\widehat{S}$  satisfying properties  $(\iota)$  and  $(\nu)$  below.*

**Theorem B** *Let  $S$  be a ruled surface with negative normal bundle inside a smooth threefold, so that  $S$  is ruled over a curve of positive genus. Fix a good polarization  $R$  on  $S$ . Let  $E = \{E_n\}$  be a vector bundle on  $\widehat{S}$  such that  $E|_S$  is  $R$ -stable. Then the local moduli space of  $E$  is finite dimensional and smooth. Moreover,  $E$  is a flat limit of a flat family of vector bundles on  $\widehat{S}$  satisfying properties  $(\iota)$  and  $(\nu)$  below.*

Let  $r := \text{rank}(E)$  and  $d := \text{deg}(E)$ . The general element  $G = \{G_n\}$  of the family has the following behavior.

$(\nu)$  *If  $d = ar - x$ ,  $0 < x < r$ , then the general element  $G$  of the family is a vector bundle such that the restriction of  $G_1$  to a general fiber  $D$  of  $u$  has splitting type  $(a, \dots, a, a - 1, \dots, a - 1)$ , and in this case*

$$G|_{\widehat{D}} \simeq \mathcal{O}_{\widehat{D}}(a)^{\oplus(r-x)} \oplus \mathcal{O}_{\widehat{D}}(a-1)^{\oplus x}.$$

(u) If  $d = ra$ , then the general element  $G$  of the family is a vector bundle such that the restriction of  $G_1$  to a general fiber  $D$  of  $u$  has splitting type  $(a, \dots, a)$  and in this case

$$G|_{\widehat{D}} \simeq \mathcal{O}_{\widehat{D}}(a)^{\oplus r}$$

but there exists a finite number of jumping fibers  $D'$  where  $G|_{D'}$  has splitting type  $(a+1, a, \dots, a, a-1)$ .

For a bundle over a Hirzebruch surface we calculate the number of such jumping fibers.

**Theorem C** Let  $z$  be number of jumping fibers of  $G$ . Set  $E = G(-a\mathbf{h})$  and  $m := \deg(u_*E)$ . Then

$$z = c_2(E) = c_2(G) - a(r-1)c_1(G) \cdot \mathbf{h} - ea^2r(r-1)/2$$

and

$$m = c_1(u_*E) = -z + c_1(G) \cdot \mathbf{h} + rae.$$

In section 2 we recall some basic concepts of deformation theory. In section 3 we consider bundles on a neighborhood of a Hirzebruch surface and prove Theorems A and C. In section 4 we consider bundles on a neighborhood of a surface ruled over a curve of higher genus and prove Theorem B.

## 2 Background material on deformations

In this paper we work only over  $\mathbb{C}$ . The basic material on the deformation theory appearing in this section is taken from Seshadri [?]. Let  $X$  be a scheme over and algebraically closed field  $k$ . Let  $R$  be a complete local ring such that  $R/m_R = k$ ,  $m_R$  the maximal ideal of  $R$  and  $R_n = R/m_R^n$ .

**Definition 2.1** A *deformation*  $Y$  of  $X$  parametrized by a scheme  $T$  with base point  $t_0$  consists of

1. a morphism  $Y \rightarrow T$  which is flat and of finite type
2. a closed point  $t_0 \in T$ , and an isomorphism  $Y_{t_0} \xrightarrow{\sim} X$ , where  $Y_{t_0} = Y \times_T k(t_0)$  is the fiber over  $t_0$ .

**Definition 2.2** A *formal deformation*  $X_R$  of  $X$  is a sequence  $\{X_n\}$  such that

1.  $X_n = X_{R_n}$  where  $X_{R_n}$  is a deformation of  $X$  over  $R_n$
2. we are given a compatible sequence of isomorphisms  $X_n \otimes_{R_n} R_{n-1} \rightarrow X_{n-1}$  for any  $n$ .

**Definition 2.3** Let  $A$  be a finite dimensional local  $k$ -algebra. Then, giving a  $k$ -algebra homomorphism  $\phi: R \rightarrow A$  is equivalent to giving a compatible sequence of homomorphisms  $\phi_n: R_n \rightarrow A$  for  $n \gg 0$ . It follows that, given a formal deformation  $X_R$  of  $X$  and a homomorphism  $\phi: R \rightarrow A$ ,  $X_n \otimes_{R_n} A$  is the same up to isomorphisms for  $n \gg 0$ . We define this to be  $X_R \otimes A$ . It is a deformation of  $X$  over  $A$ , called the *base change* of  $X_R$  by  $\text{Spec } A \rightarrow \text{Spec } X$ .

**Definition 2.4** Let  $F$  and  $G$  be the functors defined by

1.  $F(A) =$  isomorphism classes of deformations  $X_A$  over  $A$
2.  $G(A) = \text{Hom}_k(R, A)$ .

We get a morphism of functors  $j: G \rightarrow F$  defined by  $\phi \in \text{Hom}_k(R, A) \mapsto X_R \otimes A$ . A formal deformation  $X_R$  of  $X$  is said to be *versal* if the functor  $j$  is formally smooth. ([?] p.271)

More generally, one can define similarly the concept of versal deformation for a covariant functor  $F$  with  $F(k) =$  a single point. Schlessinger gave conditions for the existence of versal deformations of a functor for  $F$ . Moreover, Artin's algebraization theorem says that Schlessinger's conditions together with effectiveness imply the existence of an algebraic deformation space for  $F$ . For details see [?] and [?] .

**Remark 2.5** In the case of deformations of  $X$  algebraization means that there exists a scheme  $Y$  over  $S$  flat and of finite type, with base point  $s_0$  such that  $\widehat{\mathcal{O}}_{S, s_0} = R$  and  $Y \otimes R_n = X_n$ . The conditions for algebraization are satisfied for deformations of vector bundles over a complete algebraic scheme (see [?] thm 2.3).

**Definition 2.6** The germ of  $Y$  at  $s_0 \in S$  is determined up to isomorphism and we call it the *local moduli space* of  $X$ . (Here germ means  $\text{Spf } \mathcal{O}_{Y, s_0}$ , i.e. a compatible sequence of spectra of rings over Artinian rings). When a deformation of  $X$  is considered only on a germ at a point  $s_0$  we call it a *local deformation* of  $X$ .

In this paper we construct formal deformations of vector bundles. An application of Artin's algebraization theorem then implies the existence of algebraic deformations. We restate definition 2.1 for the case of vector bundles (definitions 2.2 and 2.4 can be repeated with  $E$  in place of  $X$  and  $\overline{E}$  in place of  $Y$  and we get the concepts of formal, versal and local deformations as well as local moduli for vector bundles).

**Definition 2.7** Let  $E$  be a vector bundle over a scheme  $X$ . A *deformation*  $\overline{E}$  of  $E$  parametrized by a scheme  $T$  with base point  $t_0$  consists of

1. a vector bundle  $\overline{E} \rightarrow X \times T$
2. a closed point  $t_0 \in T$ , and an isomorphism  $\overline{E}|_{X \times t_0} \xrightarrow{\sim} E$ .

**Definition 2.8** Let  $X_R = \{X_n\}$  be a formal deformation of  $X$ . A *vector bundle*  $E_R$  on  $X_R$  is a compatible sequence of vector bundles on each  $X_n$ . A *deformation* of  $E_R$  is given by a compatible sequence of deformations for each  $E_n$ .

**Remark 2.9** We say that the local moduli space of  $E$  is *smooth* if the germ of  $\overline{E}$  at  $X \times t_0$  is regular. In order to check that the local moduli space of  $E$  is smooth it suffices to check formal smoothness (cf. [?] remark 2.4). Obstructions for smoothness are in  $H^2(X, \text{End}(E))$ , and if this group vanishes, we say that deformations of  $E$  are *unobstructed*. It follows that the criterion for smoothness is that  $H^2(X, \text{End}(E)) = 0$ .

### 3 Bundles on neighborhood of a Hirzebruch surface

Let  $S$  be a ruled surface inside a smooth threefold  $W$ . Let  $V$  be either a neighborhood of  $S$  in  $W$  in the smooth topology, or the germ of  $W$  around  $S$ , and let  $\widehat{S}$  be the formal completion of  $S$  in  $V$ . In this section we consider the case when  $S = \Sigma_e$ ,  $e \geq 0$ , is a Hirzebruch surface. If  $e = 0$  then  $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$  and hence the two projections  $f_1: S \rightarrow \mathbb{P}^1$  (resp.  $f_2: S \rightarrow \mathbb{P}^1$ ) on the first (resp. second) factor define two rulings of  $S$ . We use only the first ruling and set  $u := f_1$ . If  $e > 0$ , then the surface  $\Sigma_e$  has a unique ruling  $u: \Sigma_e \rightarrow \mathbb{P}^1$ . Call  $f$  any fiber of  $u$ . Let  $h$  be a section of  $u$  with minimal self-intersection. We denote by  $\mathbf{f}$  (resp.  $\mathbf{h}$ ) the class of  $f$  (resp.  $h$ ) in  $\text{Pic}(S)$ . Thus  $\mathbf{f}$  and  $\mathbf{h}$  form a

basis for  $\text{Pic}(S) \simeq \mathbb{Z} \oplus \mathbb{Z}$  and have intersection numbers  $\mathbf{h}^2 = -e$ ,  $\mathbf{h} \cdot \mathbf{f} = 1$  and  $\mathbf{f}^2 = 0$ . The canonical line bundle of  $S$  is isomorphic to  $\mathcal{O}_S(-2\mathbf{h} - (e+2)\mathbf{f})$ . Let  $t$  and  $s$  be the integers such that  $I/I^2 \simeq \mathcal{O}_S(t\mathbf{h} + s\mathbf{f})$ , where  $I$  is the ideal defining  $V$  in  $W$ . We assume  $t > 0$  and  $s > et$ , that is, we assume that  $I/I^2$  is ample. For every integer  $n \geq 1$  we have  $I^n/I^{n+1} \simeq S^n(I/I^2) \simeq \mathcal{O}_S(n t \mathbf{h} + n s \mathbf{f})$ . Thus  $h^1(S, I^n/I^{n+1}) = h^2(S, I^n/I^{n+1}) = 0$  for all  $n \geq 1$ .

The vector spaces of regular functions on  $V$  and of formal functions on  $\widehat{S}$  are infinite dimensional. We consider vector bundles  $E = \{E_n\}$  on  $\widehat{S}$  such that  $E|_S$  is simple, that is, such that  $h^0(S, \text{End}(E|_S)) = 1$ . In other words, we require  $h^0(S, \text{ad}(E|_S)) = 0$ . For all integers  $n \geq 0$  we have the exact sequence

$$0 \rightarrow I^n/I^{n+1} \rightarrow \mathcal{O}_{S(n+1)} \rightarrow \mathcal{O}_{S(n)} \rightarrow 0 \quad (1)$$

For every vector bundle  $E = \{E_n\}$  on  $\widehat{S}$  we have the exact sequences

$$0 \rightarrow E_0 \otimes I^n/I^{n+1} \rightarrow E_{n+1} \rightarrow E_n \rightarrow 0 \quad (2)$$

obtained from (1) by tensoring with  $E_{n+1}$ . Take a vector bundle  $G = \{G_n\}$  on  $\widehat{S}$ . For every integer  $n \geq 0$ , set  $E_n = \text{ad}(G_n)$ , where the  $\text{ad}$  and  $\text{Hom}$  functors are computed with respect to  $\mathcal{O}_{S(n)}$ . By the long exact sequence in cohomology derived from (2) we see that the integer  $h^0(S(n), \text{ad}(G_n))$  goes to infinity when  $n$  goes to infinity. Hence, there are no simple vector bundles on  $\widehat{S}$ .

**Lemma 3.1** *Let  $E = \{E_n\}$  be a vector bundle on  $\widehat{S}$  such that  $E|_S$  is simple. Then for all integers  $n \geq 1$  we have  $h^2(S(n), \text{End}(E_n)) = 0$ .*

*Proof.* First assume  $n = 1$ . Since  $E|_S$  is simple, we have  $h^0(S, \text{End}(E|_S) \otimes \mathcal{O}_S(-2\mathbf{h} - (e+2)\mathbf{f})) = 0$ . By Serre duality

$$h^0(S, \text{End}(E|_S) \otimes \mathcal{O}_S(-2\mathbf{h} - (e+2)\mathbf{f})) = h^2(S, \text{End}(E|_S)),$$

concluding the case  $n = 1$ . Now assume  $n \geq 2$  and that the result is true for the integer  $n - 1$ , i.e. assume  $h^2(S(n-1), \text{End}(E_{n-1})) = 0$ . Since  $\dim S(n) = 2$  we have  $h^3(S(n), A) = 0$  for every coherent analytic sheaf  $A$  on  $S(n)$ . Using (2) for the integer  $n - 1$  and the vector bundle  $\text{End}(E_n)$  together with the inductive assumption, we see that  $h^2(S(n), \text{End}(E_n)) = 0$  if  $h^2(S, \text{End}(E|_S) \otimes \mathcal{O}_S(n t \mathbf{h} + n s \mathbf{f})) = 0$ . By Serre duality we have

$$\begin{aligned} h^2(S, \text{End}(E|_S) \otimes \mathcal{O}_S(n t \mathbf{h} + n s \mathbf{f})) &= \\ h^0(S, \text{End}(E|_S) \otimes \mathcal{O}_S(-(2 + n t)\mathbf{h} - (2 + e + n s)\mathbf{f})) &= 0. \end{aligned}$$

□

**Remark 3.2** If  $F$  is a vector bundle on  $S(n)$  such that  $h^2(S(n), \text{End}(F)) = 0$ , then by Remark 2.8 the local moduli space of  $F$  is smooth and has dimension  $h^1(S(n), \text{End}(F))$ .

**Lemma 3.3** Let  $E = \{E_n\}$  be a vector bundle on  $\widehat{S}$  such that  $E|_S$  is simple. Then for all integers  $n \geq 1$  the restriction map

$$h^1(S(n+1), \text{End}(E_{n+1})) \rightarrow h^1(S(n), \text{End}(E_n))$$

is surjective.

*Proof.* As in the proof of Lemma ?? we obtain  $h^2(S, \text{End}(E|_S) \otimes I^n / I^{n+1}) = 0$  for every integer  $n \geq 1$ . The lemma follows from the cohomology exact sequence of (2) with the bundle  $\text{End}(E_n)$  instead of  $E_n$ .  $\square$

**Lemma 3.4** Let  $E = \{E_n\}$  be a vector bundle on  $\widehat{S}$  such that  $E|_S$  is simple. Then there exists an integer  $x$  depending only on  $E|_S$  such that for all integers  $n \geq x$  the restriction map

$$H^1(S(n+1), \text{End}(E_{n+1})) \rightarrow H^1(S(n), \text{End}(E_n))$$

is bijective.

*Proof.* By Lemma ?? it suffices to show the existence of  $x$  such that for all  $n \geq x$  the restriction map  $H^1(S(n+1), \text{End}(E_{n+1})) \rightarrow H^1(S(n), \text{End}(E_n))$  is injective. Since  $I/I^2$  is ample there exists an integer  $x$  such that for all integers  $y \geq x$  we have  $H^1(S, \text{End}(E|_S) \otimes I^y / I^{y+1}) = 0$ . Now injectivity follows from the long exact sequence of (2) with  $\text{End}(E_n)$  in place of  $E_n$ .  $\square$

**Proposition 3.5** Let  $E = \{E_n\}$  be a vector bundle on  $\widehat{S}$  such that  $E|_S$  is simple. There exists an integer  $x$  such that  $n \geq x$  implies that every local deformation of  $E_n$  lifts to a local deformation of  $E$ .

*Proof.* By Schlessinger's theorem, a hull exists for deformations of  $E_n$ , and since by lemma ??  $h^2(S(n), \text{End}(E_n)) = 0$ , the hull is smooth. Hence it is the formal spectrum of a formal power series ring  $R_n = \mathbb{C}[[x_1, \dots, x_s]]$ . Similarly, set  $R_{n+1} = \mathbb{C}[[y_1, \dots, y_r]]$  to be the formal power series ring corresponding to  $E_{n+1}$ . By lemma ?? the map  $R_{n+1} \rightarrow R_n$  induces a bijection at tangent level, and it follows from the formal inverse function theorem, that the local

deformations of  $E(n+1)$  and  $E(n)$  are isomorphic for all  $n \geq x$  therefore they determine a local deformation of  $E$ .  $\square$

The following property of bundles on  $\mathbb{P}^1$  is well known, but we were not able to find it in the literature.

**Lemma 3.6** *Every vector bundle on  $\mathbb{P}^1$  is the flat limit of a flat family of rigid vector bundles.*

*Proof.* Let  $0 \leq c < r$  be integers and let  $A = \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus(r-c)} \oplus \mathcal{O}_{\mathbb{P}^1}(a-1)^{\oplus c}$ . This is a rigid bundle. We show that all bundles on  $\mathbb{P}^1$  with rank  $r$  and degree  $ra - c$  deform to  $A$ . Fix any such bundle  $B$  with splitting type  $b_1 \geq \dots \geq b_r$ ; we may assume  $b_r \leq b_1 - 2$ , otherwise  $B = A$ . Set  $B' = \bigoplus_{b_i \leq a-1} \mathcal{O}_{\mathbb{P}^1}(b_i)$  and  $B'' = \bigoplus_{b_i > a-1} \mathcal{O}_{\mathbb{P}^1}(b_i)$ . By Shatz [?] Proposition 1, there is an exact sequence

$$0 \rightarrow B' \rightarrow A \rightarrow B'' \rightarrow 0.$$

Call  $\mathbf{e}$  the extension class of this sequence. For all nonzero scalars  $\lambda$  the extension  $\lambda \mathbf{e}$  has  $A$  as middle term, whereas for  $\lambda = 0$  the corresponding extension has middle term  $B$ . This gives a flat specialization to  $B$  of a family of vector bundles isomorphic to  $A$ .  $\square$

**Lemma 3.7** *Let  $F$  be a simple rank  $r$  vector bundle on  $S$ . Fix a fiber  $D$  of  $u$ . Then the local moduli space of the vector bundle  $F|_D$  is smooth and of dimension  $h^1(S, \text{End}(F|_D))$ . The local moduli space of  $F$  on  $S$  is smooth and of dimension  $h^1(S, \text{End}(F))$ .*

*Proof.* Since  $\dim(D) = 1$ , we have  $H^2(S, \text{End}(F|_D)) = 0$  and hence the local moduli space of the vector bundle  $F|_D$  is smooth and of dimension  $H^1(S, \text{End}(F|_D))$ . Consider the exact sequence

$$0 \rightarrow \text{End}(F)(-D) \rightarrow \text{End}(F) \rightarrow \text{End}(F|_D) \rightarrow 0. \quad (3)$$

By Serre duality we have

$$h^2(S, \text{End}(F)(-D)) = h^0(S, \text{End}(F)(-D) \otimes \mathcal{O}_S(-2\mathbf{h} - (1+e)\mathbf{f})).$$

Since  $h^0(S, \text{End}(F)) = 1$ , we have that  $h^0(\text{End}(F)(-D) \otimes \mathcal{O}_S(-2\mathbf{h} - (1+e)\mathbf{f})) = 0$  and hence  $h^2(\text{End}(F)(-D)) = 0$ . It follows that  $h^2(S, \text{End}(F)) = 0$ , we obtain the result for the local moduli space of  $F$ .  $\square$

The following observation was inspired by [?] Lemmas 2 and 3, and their use in [?].



**Proposition 3.8** *Let  $F$  be a simple vector bundle on  $S$ . Then  $F$  is a flat limit of a flat family of vector bundles, whose restriction to  $D$  is rigid.*

*Proof.* Since  $h^2(\text{End}(F)(-D)) = 0$  as shown in the proof of lemma ??, by the exact sequence (3) we obtain that the restriction map

$$\gamma: H^1(S, \text{End}(F)) \rightarrow H^1(S, \text{End}(F|_D)) \quad (4)$$

is surjective. The surjectivity of  $\gamma$  means that every local deformation of  $F|_D$  may be lifted to a local deformation of  $F$ . Thus, we obtain from lemma ?? that  $F$  is a flat limit of a flat family of vector bundles, whose restriction to  $D$  is rigid.  $\square$

### 3.1 Case 1: the rank does not divide the degree

Let  $F$  be a simple vector bundle on  $S$  whose rank  $r \geq 2$  does not divide the degree. Write  $\det(F) \cdot \mathcal{O}_S(\mathbf{h}) = ar - x$ , with  $0 < x < r$ . We construct the local moduli space of  $F$ .

Take any fiber  $K$  of  $u$ . If  $F|_K$  is not rigid, then by [?] Lemma 2, it is of special type among deformations of bundles on  $\mathbb{P}^1$ ; such types occurring in a subset of codimension at least 3 inside a complete family. The following trick we stole from [?] and it was the starting point of our paper. The surjectivity of  $\gamma$  in (4) implies that the general member of this family in Proposition ?? has, as restriction to any fiber, a general deformation of  $F|_K$ , i.e. the only rigid vector bundle on  $\mathbb{P}^1$  with degree  $ar - x$ , i.e. the bundle on  $\mathbb{P}^1$  with splitting type  $(a, \dots, a, a - 1, \dots, a - 1)$  (with  $a - 1$  appearing  $x$  times). Therefore there exists a flat deformation of  $F$  whose general  $G$  element has rigid restriction to all except finitely many fibers of  $u$ . This implies that  $G$  is uniform in the sense of Ishimura with respect to the ruling  $u$  (see [?]). Changing bases,  $H := u_*(G \otimes \mathcal{O}_S(-a\mathbf{h}))$  is a rank  $r - x$  vector bundle on  $\mathbb{P}^1$  and  $u^*(H)$  is a rank  $r - x$  sub-bundle of  $G \otimes \mathcal{O}_S(-a\mathbf{h})$  and

$$\frac{G \otimes \mathcal{O}_S(-a\mathbf{h})}{u^*(H)} \otimes \mathcal{O}_S((a - 1)\mathbf{h}) \simeq u^*(H)$$

for some rank  $x$  vector bundle  $M$  on  $\mathbb{P}^1$ . Thus we have an exact sequence

$$0 \rightarrow u^*(H) \otimes \mathcal{O}_S(a\mathbf{h}) \rightarrow G \rightarrow u^*(M) \otimes \mathcal{O}_S((a - 1)\mathbf{h}) \rightarrow 0. \quad (5)$$

The Chern classes of  $G$  are uniquely determined by the integers  $e, a, x, \deg(H)$  and  $\deg(M)$ . Conversely, the integers  $\deg(M)$  and  $\deg(H)$  are uniquely determined by  $r, a, x$  and the Chern classes of  $G$ . We set the shorthands

$$\bar{M} := u^*(M) \otimes \mathcal{O}_S((a-1)\mathbf{h}),$$

$$\bar{H} := u^*(H) \otimes \mathcal{O}_S(a\mathbf{h}),$$

$$K := \mathcal{O}_S(-2\mathbf{h} - (e+2)\mathbf{f}).$$

By Serre duality

$$h^2(S, \text{Hom}(\bar{M}, \bar{H})) = h^0(S, \text{Hom}(\bar{M}, \bar{H} \otimes K)).$$

The restriction of  $\text{Hom}(\bar{M}, \bar{H} \otimes K)$  to any fiber of  $u$  is a direct sum of line bundles of degree at most  $-3$ . Thus

$$h^2(S, \text{Hom}(\bar{M}, \bar{H})) = 0.$$

By Riemann Roch, the integer  $h^1(S, \text{Hom}(\bar{M}, \bar{H}))$  depends only on the numerical data  $e, r, x, a, \deg(H)$ , and  $\deg(M)$  but not on the specific choice of the vector bundles  $M$  and  $H$ . From the properties of the universal Ext functor (see [?]) it follows that, corresponding to any family of vector bundles  $\{M_t\}_{t \in T}$  and  $\{H_t\}_{t \in T}$  parametrized by an irreducible variety  $T$ , we obtain a flat family  $\mathbf{V} = \{G_t\}_{t \in T}$  of middle terms of extensions (4) parametrized by  $T$  with each  $G_t$  a rank  $r$  vector bundle on  $T$ . Since  $T$  is assumed to be irreducible,  $\mathbf{V}$  is irreducible.

We choose as the parameter space  $T$  the product of a versal deformation space of  $M$  and a versal deformation space of  $H$ . Such spaces are irreducible, smooth and of dimension respectively  $h^1(\mathbb{P}^1, \text{End}(M))$  and  $h^1(\mathbb{P}^1, \text{End}(H))$ . With this choice of  $T$ , for a general  $t \in T$  the vector bundles  $M$  and  $H$  are rigid and the general vector bundle  $G_t$  of the family is an extension of the form (4) with  $M$  and  $H$  rigid. The set of all such extensions is a vector space, whose dimension depends only on the numerical data  $r, c_1(G)$ , and  $c_2(G)$ . Combining with (4) we obtain that any simple rank  $r$  vector bundle  $F$  with degree  $ar - x$  is the flat limit of a family of simple vector bundles  $G_t$  arising from the family just described, that is, from an extension (4) with  $M$  and  $H$  rigid.

We now extend the construction of the flat family to bundles on  $\widehat{S}$ . Let  $D \simeq \mathbb{P}^1$  be a fiber of  $u$  and  $J$  the ideal sheaf of  $D$  in  $V$  or  $\widehat{S}$ . Let  $\widehat{D} = \{D(n)\}$

be the formal completion of  $D$  in  $\widehat{S}$ . Hence  $D(n)$  has  $J^n$  as ideal sheaf in  $\widehat{S}$  (or in  $V$ ) and  $D(0) = D$ . We have  $J/J^2 \simeq \mathcal{O}_D(t) \otimes \mathcal{O}_D$  and  $J^n/J^{n+1} \simeq S^n(J/J^2)$  for every  $n \geq 1$ . Let  $A = \{A_n\}$  be a rank  $r$  vector bundle on  $\widehat{S}$ . For every integer  $n \geq 1$  we have an exact sequence

$$0 \rightarrow J^n/J^{n+1} \otimes A_1 \rightarrow A_{n+1} \rightarrow A_n \rightarrow 0. \quad (6)$$

**Lemma 3.9** *For every  $n \geq 1$  the restriction maps  $\rho_n: \text{Pic}(D(n)) \rightarrow \text{Pic}(D)$  and  $\rho: \text{Pic}(\widehat{D}) \rightarrow \text{Pic}(D)$  are bijective.*

*Proof.* Since  $h^2(D, J^n/J^{n+1}) = 0$  for every  $n \geq 1$ , to obtain the surjectivity of  $\rho_n$  it is sufficient to copy [?], part 2 of the proof of theorem 3.1 on page 179. The last assertion follows from the bijectivity of all maps  $\rho_n$ , just by the definition of line bundle on a formal scheme.  $\square$

The proof of Lemma ?? gives without any modification the analogous result with  $S$  in place of  $D$ .

**Lemma 3.10** *For every  $n \geq 1$  the restriction maps  $\rho_n: \text{Pic}(S(n)) \rightarrow \text{Pic}(S)$  and  $\rho: \text{Pic}(\widehat{S}) \rightarrow \text{Pic}(S)$  are bijective.*

By Lemma ?? we can write  $\mathcal{O}_{D(n)}(a)$  (resp.  $\mathcal{O}_{\widehat{D}}(a)$ ) for the unique line bundle on  $D(n)$  (resp.  $\widehat{D}$ ) whose restriction to  $D$  has degree  $a$ . By Lemma ?? we can write  $\mathcal{O}_{S(n)}(a, b)$  (resp.  $\mathcal{O}_{\widehat{S}}(a, b)$ ) for the unique (up to isomorphism) line bundle on  $S(n)$  (resp.  $\widehat{S}$ ) whose restriction to  $S$  is isomorphic to  $\mathcal{O}_S(a, b)$ .

**Proposition 3.11** *If the restriction of  $G_1$  to a fiber  $D$  has splitting type  $(a, \dots, a, a-1, \dots, a-1)$ , with  $a-1$  appearing  $x$  times, then*

$$G|_{\widehat{D}} \simeq \mathcal{O}_{\widehat{D}}(a)^{\oplus(r-x)} \oplus \mathcal{O}_{\widehat{D}}(a-1)^{\oplus x}.$$

*Proof.* Let  $J$  be the ideal sheaf of  $D$  in  $V$  (or  $\widehat{S}$ ). We have  $J/J^2 \simeq \mathcal{O}_D(t) \oplus \mathcal{O}_D$ . Since by our assumptions  $t > 0$ , then for every integer  $n \geq 1$  the sheaf of  $\mathcal{O}_D$ -modules  $J^n/J^{n+1} \simeq S^n(J/J^2)$  is the direct sum of  $n+1$  line bundles on  $D$  with nonnegative degree. Set

$$A_n := \text{Hom}(\mathcal{O}_{D(n)}(a)^{\oplus(r-x)} \oplus \mathcal{O}_{D(n)}(a-1)^{\oplus x}, G_n).$$

Thus  $\{A_n\}$  is a rank  $r^2$  vector bundle on  $\widehat{S}$  and  $h^1(D, A_1) = 0$ . Fix an integer  $n \geq 1$  and assume  $G_n \simeq \mathcal{O}_{D(n)}(a)^{\oplus(r-x)} \oplus \mathcal{O}_{D(n)}(a-1)^{\oplus x}$ . Fix  $m_n \in H^0(D(n), A_n)$  with  $m_n$  invertible. We have  $H^1(D, J^n/J^{n+1} \otimes (a-1)) = 0$ . Thus by (5) we may lift  $m_n$  to  $m_{n+1} \in H^0(D(n+1), A_{n+1})$  with  $m_{n+1}|_{D(n)} = m_n$ . By Nakayama's lemma  $m_n$  is invertible.  $\square$

### 3.2 Case 2: the rank divides the degree

Fix integers  $r$  and  $a$  with  $r \geq 2$ . Let  $F$  be a simple rank  $r$  vector bundle on  $S$  such that  $\det(F) \cdot \mathcal{O}_S(\mathbf{h}) = ar$ , i.e., such that the restriction of to any fiber of  $u$  has degree  $d = ar$ . Since  $F$  is simple, by ?? we have that  $F$  is a flat limit of a flat family of vector bundles on  $S$ . The general element of this family is a vector bundle  $G$  whose restriction to a general fiber of  $u$  has splitting type  $(a, \dots, a)$  (i.e. it is isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r}$ ) but for which there are finitely many (say  $z$ ) fibers of  $u$  such that the restriction of  $G$  to each of these  $z$  fibers has splitting type  $(a+1, a, \dots, a, a-1)$  (i.e. it is isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(a+1) \oplus \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus(r-2)} \oplus \mathcal{O}_{\mathbb{P}^1}(a-1)$ ). The  $z$  fibers of  $u$  arising in this way are called the *jumping fibers* of  $G$ .

**Theorem C** *Let  $z$  be number of jumping fibers of  $G$ . Set  $E = G(-a\mathbf{h})$  and  $m := \deg(u_*E)$ . Then*

$$z = c_2(E) = c_2(G) - a(r-1)c_1(G) \cdot \mathbf{h} - ea^2r(r-1)/2$$

and

$$m = c_1(u_*E) = -z + c_1(G) \cdot \mathbf{h} + rae.$$

*Proof.* If  $E = G(-a\mathbf{h})$  then the restriction of  $E$  to a general fiber has splitting type  $(0, \dots, 0)$  whereas the jumping fibers of  $E$  have type  $(1, 0, \dots, 0, -1)$ . It follows that for each point  $p \in C$  such that  $u^{-1}(p)$  is a jumping fiber of  $E$  the length  $l(R^1u_*E_p(-1)) = 1$ . Hence each jumping fiber has multiplicity one and  $z = \sum_p l(R^1u_*E_p(-1)) = c_2(E)$ . By[?] Lemma 2.1  $c_2(E) = c_2(G) - a(r-1)c_1(G) \cdot \mathbf{h} - ea^2r(r-1)/2$ .

By Grothendieck–Riemann–Roch  $\text{ch}(u_*E)\text{td}(T_C) = u_*(\text{ch}(E)\text{td}(T_S))$ , which gives

$$c_1(u_*E) = -z + u_* \left( (c_1 - rah) \cdot \left( \mathbf{h} + \frac{e+2}{2} \mathbf{f} \right) \right)$$

and it follows that  $m = c_1(u_*E) = -z + c_1(G) \cdot \mathbf{h} + rae$ .  $\square$

Changing bases, the coherent sheaf  $H := u_*(G \otimes \mathcal{O}_S(-a\mathbf{h}))$  is a rank  $r$  vector bundle on  $\mathbb{P}^1$ . The sheaf  $u^*(H)$  is a rank  $r$  subsheaf of  $G$  with  $G/(u^*(H) \otimes \mathcal{O}_S(a\mathbf{h}))$  supported at the jumping fibers of  $G$ . The integers  $e, r, a$  and the Chern classes of  $G$  are uniquely determine the integers  $z$  and

$\deg(H)$ . Conversely, the Chern classes of  $G$  are uniquely determined by the integers  $e, r, a, \deg(H)$  and  $z$ .  $G$  is a flat limit of a family  $\{G_\alpha\}$  of simple vector bundles with the same properties of splitting type with respect to the fibers of  $u$ , but with the added condition that  $H_\alpha = u^*(G_\alpha(\mathbf{a}\mathbf{h}))$  is rigid. Hence for such  $G_\alpha$  the vector bundle  $H_\alpha$  is uniquely determined by the integers  $r$  and  $\deg(H)$ . Only the position of the  $z$  points of  $G_\alpha/u^*(H_\alpha)(\mathbf{a}\mathbf{h})$  and the extension class of the sheaf  $G_\alpha/u^*(H_\alpha)(\mathbf{a}\mathbf{h})$  by the sheaf  $u^*(H_\alpha)(\mathbf{a}\mathbf{h})$  depend on  $\alpha$ .

**Proposition 3.12** *Let  $\{G_n\}$  be a rank  $r$  vector bundle on  $\widehat{S}$  such that  $G_1$  is simple. Let  $D$  be a fiber of  $u$  such that  $G_1|_D$  has splitting type  $(a, \dots, a)$ . Then  $G|_{\widehat{D}} \simeq \mathcal{O}_{\widehat{D}}(a)^{\oplus r}$ .*

*Proof.* Twisting by the line bundle  $\mathcal{O}_{\widehat{D}}(a)$  we reduce to the case  $a = 0$ . Let  $J$  be the ideal sheaf of  $D$  in  $\widehat{S}$ . We have  $J/J^2 \simeq \mathcal{O}_D(t) \oplus \mathcal{O}_D$  and for every integer  $n \geq 1$  the  $\mathcal{O}_D$ -sheaf  $J^n/J^{n+1} \simeq S^n(J/J^2)$  is the direct sum of  $n + 1$  line bundles on  $D$  with nonnegative degree. Set

$$A_n := \text{Hom}(\mathcal{O}_D(n)^{\oplus r}, G_n).$$

Thus  $\{A_n\}$  is a rank  $r^2$  vector bundle on  $\widehat{S}$  and  $A_1$  is trivial. Fix an integer  $n \geq 1$  and assume  $G_n$  is trivial. Fix a trivialization of  $G_n$ , i.e. fix  $m_n \in H^0(D(n), A_n)$  with  $m_n$  invertible. Then  $H^1(D, J^n/J^{n+1} \otimes A_1) = 0$  and by (5) we may lift  $m_n$  to  $m_{n+1} \in H^0(D(n+1), A_{n+1})$  with  $m_{n+1}|_{D(n)} = m_n$ . By Nakayama's lemma  $m_n$  is invertible.  $\square$

*Proof of Theorem A:* For smoothness of the deformation space apply Lemma ?? and Remark ?. Properties (i) and (ii) follow from Propositions ?? and ?.  $\square$

## 4 Bundles on a neighborhood of a surface ruled over a curve of higher genus

We re-study the theory just done, now for the case in which the divisor  $S$  is ruled over a smooth curve  $C$  of genus  $g > 0$ .  $S := \mathbb{P}(B)$  is the projectivization of a rank two vector bundle  $B$  over  $C$ . Let  $u: S \rightarrow C$  be the projection. Fix a section,  $h$ , of  $S$  with minimal self-intersection and set  $e := -h^2$ . Denote

by  $\mathbf{h}$  the class of  $h$ . By a theorem of M. Nagata and C. Segre  $e \geq -q$  and every integer  $\geq -q$  occurs for some rank two vector bundle  $B$  on  $C$  (see the introduction of [?]).  $\text{Pic}(S) \simeq u^*(\text{Pic}(C)) \oplus \mathbb{Z}[\mathbf{h}]$  and  $u^*(M) \cdot \mathbf{h} = \deg(M)$  and  $u^*(M) \cdot u^*(H) = 0$  for all  $M, H \in \text{Pic}(C)$ . The ruling  $u$  induces an isomorphism between  $h$  and  $C$  and we use this isomorphism to identify the normal bundle of  $h$  in  $S$  with a line bundle  $N$  on  $C$  with  $\deg(N) = -e$ . The canonical line bundle of  $S$  is isomorphic to  $u^*(w_C \otimes N)(-2\mathbf{h})$ . We assume that  $S$  is contained in a smooth threefold  $W$ . We use the letter  $V$  to denote either a small neighborhood of  $S$  in the smooth topology in  $W$ , or else the germ of  $W$  around  $V$ . Let  $\widehat{S}$  be the formal completion of  $S$  in  $V$  and  $I$  be the ideal sheaf of  $S$  in  $V$  or in  $\widehat{S}$ . Define  $\mathcal{A} \in \text{Pic}(C)$  and  $t \in \mathbb{Z}$  by the relation  $I/I^2 \simeq u^*(\mathcal{A})(t\mathbf{h})$ . We assume  $t > 0$  and  $\deg(\mathcal{A}) > 2q - 2 + |e|$ . By Riemann Roch,  $h^0(C, \mathcal{A}) > 0$  and  $h^0(C, \mathcal{A} \otimes N) > 0$ .

**Definition 4.1** A *good polarization* of a ruled surface  $u: S \rightarrow C$  is an ample divisor  $R \in \text{Pic}(S)$  such that  $R \cdot w_S + R \cdot D < 0$  for every fiber  $D$  of  $u$ .

**Remark 4.2** Good polarizations exist and occur quite frequently. For instance, choose any ample divisor  $H \in \text{Pic}(S)$ . Then, for any fiber  $D$  of  $u$  we have  $D \cdot w_S = -2$ , and  $H(tD) \cdot w_S + H(tD) \cdot D < 0$  for every integer  $t \gg 0$ . Furthermore  $H(tD)$  is ample for every  $t \geq 0$  because  $D$  is numerically effective. Thus, for  $t \gg 0$  the line bundle  $H(tD)$  is a good polarization.

Our definition of good polarization is exactly the definition for which Lemma ?? and Proposition ?? work for  $q > 0$  and for any  $R$ -stable vector bundle on  $S$ . In what follows we assume that  $R$  is a good polarization of  $S$ .

**Lemma 4.3** Let  $E = \{E_n\}$  be a vector bundle on  $\widehat{S}$  such that  $E|_S$  is  $R$ -stable in the sense of Mumford and Takemoto. Then,  $h^2(S(n), \text{End}(E_n)) = 0$  for all  $n \geq 1$ .

*Proof.* First assume  $n = 1$ . Since  $E|_S$  is  $R$ -stable and  $R \cdot w_S < 0$ , we have  $h^0(S, \text{Hom}(E|_S, E|_S \otimes w_S)) = 0$ . By Serre duality,

$$h^0(S, \text{Hom}(E|_S, E|_S \otimes w_S)) = h^2(S, \text{End}(E|_S)),$$

concluding the case  $n = 1$ . Now assume  $n \geq 2$  and that the result is true for the integer  $n - 1$ , i.e., assume  $h^2(S(n - 1), \text{End}(E_{n-1})) = 0$ . Define  $\mathcal{A}$  by  $I/I^2 \simeq u^*(\mathcal{A})(t\mathbf{h})$  as in the introduction of this section and set  $\mathcal{N} := I/I^2$ .

Since  $\dim(S(n)) = 2$  we have  $h^3(S(n), A) = 0$  for every coherent analytic sheaf  $A$  on  $S(n)$ . Using (2) for the integer  $n - 1$  and the vector bundle  $\text{End}(E_n)$  instead of  $E_n$ , together with the inductive assumption, we see that  $h^2(S(n), \text{End}(E_n)) = 0$  if  $h^2(S, \text{End}(E|_S) \otimes \mathcal{N}^{\otimes n}) = 0$ . Since  $h^0(C, \mathcal{A}^{\otimes n}) > 0$  and  $R$  is a good polarization, we have  $R \cdot w_S \otimes \mathcal{N}^{\otimes n} < 0$ . By Serre duality,

$$h^2(S, \text{End}(E|_S) \otimes \mathcal{N}^{\otimes n}) = h^0(S, \text{Hom}(E|_S, E|_S \otimes w_S \otimes (\mathcal{N}^*)^{\otimes n}))$$

which vanishes, by the  $R$ -stability of  $E|_S$ .  $\square$

The proof of Lemma ?? works verbatim in this situation just assuming that  $F$  is  $R$ -stable for some good polarization  $R$  on  $S$ . Lemmas ?? and ?? work verbatim. Therefore, we can write  $\mathcal{O}_{\widehat{D}}(a)$  for the unique line bundle on  $\widehat{D}$  whose restriction to  $D$  has degree  $a$  and we write  $\mathcal{O}_{\widehat{S}}u^*(M)(\mathbf{th})$  for the unique (up to isomorphism) line bundle on  $\widehat{S}$  whose restriction to  $S$  is isomorphic to  $\mathcal{O}_Su^*(M)(\mathbf{th})$ . We may now re-state 2.9 in the situation.

**Proposition 4.4** *Fix integers  $r, a$  and  $x$  with  $r \geq 2$  and  $0 < x < r$ . Let  $G = \{G_n\}$  be a vector bundle on  $\widehat{S}$  such that the restriction of  $G_1$  to a fiber  $D$  of  $u$  has splitting type  $(a, \dots, a, a - 1, \dots, a - 1)$ , with  $a - 1$  appearing  $x$  times. Then  $G|_{\widehat{D}} \simeq \mathcal{O}_{\widehat{D}}(a)^{\oplus(r-x)} \oplus \mathcal{O}_{\widehat{D}}(a - 1)^{\oplus x}$ .*

Let  $F$  be an  $R$ -stable rank  $r$  vector bundle on  $S$  such that  $\det(F) \cdot \mathcal{O}_S(\mathbf{h}) = ar$ , that is, such that the restriction of to any fiber of  $u$  has degree  $ar$ . Then ?? and ?? hold with  $E|_S$   $R$ -stable in place of  $E|_S$  simple, we obtain.

**Proposition 4.5** *Fix integers  $r, a$  with  $r \geq a$ . Let  $\{G_n\}$  be a rank  $r$  vector bundle on  $\widehat{S}$  such that  $G_1$  is  $R$ -stable. Let  $D$  be a fiber of  $u$  such that  $G_1|_D \simeq \mathcal{O}_D(a)^{\oplus r}$ . Then  $G|_{\widehat{D}} \simeq \mathcal{O}_{\widehat{D}}(a)^{\oplus r}$ .*

*Proof of Theorem B:* For smoothness of the deformation space apply Lemma ?? and Remark ?. Properties  $(\iota)$  and  $(\iota)$  follow from Propositions ?? and ??.  $\square$

## References

- [1] M. Artin *Algebraization of formal moduli. I, Global Analysis (Papers in Honor of K. Kodaira)* Univ. Tokyo Press, Tokyo (1969) pp. 21–71.

- [2] *J. Brun and A. Hirshowitz*, Droites de saut des fibrés stables de rang élevé sur  $\mathbb{P}^2$ , *Math. Z.* **181** (1982), 171-178.
- [3] *G. Ellingsrud*, Sur l'irréductibilité du module des fibrés stables sur  $\mathbb{P}^2$ , *Math. Z.* **182** (1983), 189-192.
- [4] *R. Hartshorne*, Ample subvarieties of algebraic varieties, *Lect. Notes in Math.* **156**, Springer Verlag (1970).
- [5] *R. Hartshorne*, Stable Reflexive Sheaves, *Math. Ann.* **254** (1980), 121-176
- [6] *S. Ishimura*, On  $\pi$ -uniform vector bundles on curves, *Nagoya Math. J.* **43** (1971), 71-89.
- [7] *H. Lange*, Universal families of extensions, *J. Algebra* **83** (1983), 101-112.
- [8] *H. Lange and M. S. Narasimhan*, Maximal sub-bundles of rank two vector bundles on curves, *Math. Ann.* **266** (1983), 55-72.
- [9] *C. S. Seshadri*, Theory of moduli, *Proc. Symp. Pure Math.* **29** (1975), 263-304.
- [10] *M. Schlessinger*, Functors of Artin rings, *Trans. Amer. Math. Soc.* **130** (1968), 208-222.
- [11] *S. S. Shatz*, On subbundles of vector bundles over  $\mathbf{P}^1$ , *J. Pure Appl. Algebra* **10** (1977/78), no. 3, 315-322 .

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