COMPACTIFICATIONS OF ADJOINT ORBITS AND THEIR HODGE DIAMONDS

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ABSTRACT. A recent theorem of [GGS2] showed that adjoint orbits of semisimple Lie algebras have the structure of symplectic Lefschetz fibrations. We investigate the behaviour of their fibrewise compactifications. Expressing adjoint orbits and fibres as affine varieties in their Lie algebra, we compactify them to projective varieties via homogenisation of the defining ideals. We find that their Hodge diamonds vary wildly according to the choice of homogenisation, and that extensions of the potential to the compactification must acquire degenerate singularities.

CONTENTS

1. HODGE DIAMONDS OF LEFSCHETZ FIBRATIONS

Given a symplectic manifold X, a symplectic Lefschetz fibration (SLF) on X is a fibration $f : X \to \mathbb{C}$ that has only Morse type singularities such that the fibres of f are symplectic submanifolds of X outside the critical set, see [Se]. We constructed a large family of new examples of noncompact SLFs in the recent paper [GGS1] and needed to compactify them to find the topological information provided by their Hodge diamonds. Our motivation – coming from mathematical physics – was to study categories of Lagrangian vanishing cycles. These play an essential role in the Homological Mirror Symmetry conjecture [Ko], where such a category appears as the Fukaya category of a Landau-Ginzburg (LG) model (that is, a Kähler manifold X equipped with

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a holomorphic function $f: X \to \mathbb{C}$ called the superpotential). SLFs are nice examples of LG models since a rigorous definition of the Fukaya category of Lagrangian vanishing cycles is known only for SLFs and not in any greater generality.

[GGS1] showed the existence of the structure of SLFs on adjoint orbits of semisimple Lie algebras. These adjoint orbits are not compact. In fact, they are isomorphic to cotangent bundles of flag varieties [GGS2]. We want to compare the behaviour of vanishing cycles on X and on its compactifications. Expressing the adjoint orbit as an algebraic variety, we homogenise its ideal to obtain a projective variety, which serves as our compactification. To obtain topological information about the compactifications, we calculate their Hodge diamonds, as well as the Hodge diamonds of the compactified fibres of the SLF. Calculating such Hodge diamonds is computationally heavy, so we used Macaulay2. Details of the computational algorithms we used appear in [CG]. Topological data for the total space X as well as for the fibres of the SLF can be read off the Hodge diamonds.

Remark 1. Choosing a compactification is in general a delicate task: a different choice of generators for the defining ideal of the orbit can result in completely different Hodge diamonds of the corresponding compactification. This happens because the homogenisation of an ideal I can change drastically if we vary the choice of generators for I (see Section 6.2).

In Section 2, we present the principal theorem that furnishes us with examples. In Section 3, we find all adjoint orbits of $\mathfrak{sl}(2,\mathbb{C})$ (up to isomorphism), and apply our compactification process to this simple case. In Section 4, we consider a more involved example of an adjoint orbit inside $\mathfrak{sl}(3,\mathbb{C})$, corresponding to the minimal flag variety, and show that any extension of the potential to the compactification of the orbit must acquire degenerate singularities, hence it would no longer remain a Lefschetz fibration. This is generalised in Section 5 to the minimal flag variety of $\mathfrak{sl}(n+1,\mathbb{C})$. We illustrate with an example in Section 6 just how delicate a tast compactification can be.

2. Lefschetz fibrations on adjoint orbits

Let H_0 be an element in the Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$, and let $\mathcal{O}(H_0)$ denote its adjoint orbit. It is proved in [GGS1] that for each regular element $H \in \mathfrak{g}$, the function $f_H: \mathcal{O}(H_0) \to \mathbb{C}$ given by $f_H(x) = \langle H, x\rangle$ gives the orbit the structure of a symplectic Lefschetz fibration. This includes the following properties for f_H:

(1) The singularities are (Hessian) nondegenerate.
(2) If $c_1, c_2 \in \mathbb{C}$ are regular values then the level manifolds $f_H^{-1}(c_1)$ and $f_H^{-1}(c_2)$ are diffeomorphic.
(3) There exists a symplectic form Ω in $\mathcal{O}(H_0)$ such that if $c \in \mathbb{C}$ is a regular value then the level manifold $f_H^{-1}(c)$ is symplectic; that is, the restriction of Ω to $f_H^{-1}(c)$ is a symplectic (nondegenerate) form.
(4) If $c \in \mathbb{C}$ is a singular value, then $f_H^{-1}(c)$ is a union of affine subspaces (contained in $\mathcal{O}(H_0)$). These subspaces are symplectic with respect to the form Ω from the previous item.

We compactify the orbit by projectivisation; that is, we homogenise the polynomials with an extra variable t to obtain a projective variety.

3. Compactification of the orbit of $\mathfrak{sl}(2,\mathbb{C})$

Inside $\mathfrak{sl}(2,\mathbb{C})$, all adjoint orbits are of the same ismomorphism type, which we now describe as an SLF with 2 critical values. In $\mathfrak{sl}(2,\mathbb{C})$, take

$$H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is regular since it has 2 distinct eigenvalues. The orbit $\mathscr{O}(H_0)$ is the set of matrices in $\mathfrak{sl}(2,\mathbb{C})$ with eigenvalues 1 and -1, which forms a submanifold of complex dimension 2 of $\mathfrak{sl}(2,\mathbb{C})$.

The Weyl group $\mathscr{W} \simeq S_2$ acts via conjugation by permutation matrices. The two singularities are thus H and $-H$.

We can also express the orbit as an affine variety embedded in $\mathbb{C}^3$. Writing a general element $A \in \mathscr{O}(H_0)$ as

$$A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix},$$

the characteristic polynomial of A is

$$-(x-\lambda)(x+\lambda) - yz = \det(A - \lambda\,\mathrm{id}) = \lambda^2 - 1,$$

the first equality being derived from explicit calculation and the second due to the fact that $\operatorname{tr} A = 0$ and $\det A = -1$. This in turn implies that the orbit $\mathscr{O}(H_0) \subset \mathfrak{sl}(2,\mathbb{C}) \simeq \mathbb{C}^3$ is an affine variety X cut out by the equation

$$x^2 + yz - 1 = 0. \tag{1}$$

We can compactify this variety by homogenising eq. 1 and embedding X into the corresponding projective variety. This gives the surface cut out by $x^2 + yz - t^2 = 0$ in $\mathbb{P}^3$. The Hodge diamond of this compactification is shown in figure 1.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

FIGURE 1. The Hodge diamond of the projectivisation of $\mathscr{O}(\mathrm{Diag}(1,-1))$.

The height function is

$$f_H(A) = \operatorname{tr} HA = \operatorname{tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = 2x.$$

Note that the two critical points belong to distinct fibres. We can also write the regular fibre (over zero) as the affine variety in $\{(y,z) \in \mathbb{C}^2\}$ cut out by the equation

$$yz - 1 = 0$$

since it must satisfy eq. 1 and $x = 0$. As with the orbit, we homogenise this equation and embed the fibre into the corresponding projective variety cut out by the equations $x = 0$ and $yz - t^2 = 0$ in $\mathbb{P}^3$. This yields the Hodge diamond shown in fig. 2. Note that these compactified fibres have no middle homology.

$$\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}$$

FIGURE 2. The Hodge diamond of the projectivisation of the regular fibre over zero, where $H = H_0 = \mathrm{Diag}(1,-1)$.

4. Smooth compactification of an $\mathfrak{sl}(3,\mathbb{C})$ orbit

The adjoint orbits of $\mathfrak{sl}(3,\mathbb{C})$ fall into one of three isomorphisms types. Here we present an SLF with 3 critical values. In $\mathfrak{sl}(3,\mathbb{C})$, consider the orbit $\mathscr{O}(H_0)$ of

$$H_0 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

under the adjoint action. We fix the element

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

to define the potential f_H. A general element $A \in \mathfrak{sl}(3,\mathbb{C})$ has the form

$$A = \begin{pmatrix} x_1 & y_1 & y_2 \\ z_1 & x_2 & y_3 \\ z_2 & z_3 & -x_1 - x_2 \end{pmatrix}. \tag{2}$$

In this example, the adjoint orbit $\mathscr{O}(H_0)$ consists of all the matrices with the minimal polynomial $(A + \mathrm{id})(A - 2\,\mathrm{id})$. So, the orbit is the affine variety cut out by the ideal I generated by the polynomial entries of $(A + \mathrm{id})(A - 2\,\mathrm{id})$. To obtain a projectivisation of X, we first homogenise its ideal I with respect to a new variable t, then take the corresponding projective variety. In this case, the projective variety $\overline{X}$ is a smooth compactification of X. We used Macaulay2 [M2] to calculate the Hodge diamonds of a compactification of the adjoint orbit $\mathscr{O}(H_0)$, obtaining:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 2 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 3 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}.$$

We now calculate the Hodge diamond of a compactified regular fibre. The potential corresponding to our choice of H is $f_H = x_1 - x_2$. The critical values of this potential are ± 3 and 0. Since all regular fibres of an SLF are isomorphic, it suffices to chose the regular value 1. We then define the regular fibre X_1 as the variety in $\mathfrak{sl}(3,\mathbb{C}) \cong \mathbb{C}^8$ corresponding to the ideal J obtained by summing I with the ideal generated by $f_H - 1$. We then homogenise J to obtain a projectivisation $\overline{X}_1$ of the regular fibre X_1. The Hodge diamond of $\overline{X}_1$ is:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

Remark 2. We used the same method to calculate the Hodge diamonds for the singular fibre over 0 and obtained the same Hodge diamond as for the regular fibres.

Remark 3. More details of this example appear in [C].

5. Generalisations and computational corollaries

We generalise our example of $\mathfrak{sl}(3,\mathbb{C})$ to $\mathfrak{sl}(n+1,\mathbb{C})$. To obtain the case where the adjoint orbit is isomorphic to the cotangent bundle of the minimal flag, we set $H_0 = \mathrm{Diag}(n,-1,\dots,-1)$ and $H = \mathrm{Diag}(1,-1,0,\dots,0)$. Then the diffeomorphism type of the adjoint orbit is given by $\mathcal{O}(H_0) \simeq T^*\mathbb{P}^n$ (see [GGS2]), and H gives the potential $x_1 - x_2$ as before. If we compactify this orbit to $\mathbb{P}^n \times (\mathbb{P}^n)^*$, then the Hodge classes of the compactification are given by $h^{p,p} = n+1-|n-p|$ and the remaining Hodge numbers are 0. An application of the Lefschetz hyperplane theorem determines all but the Hodge numbers of the middle row of the compactification of the regular fibre, and computations shows the latter are zero.

The following two corollaries follow immediately from observing the Hodge diamonds we obtained.

Corollary 1. *Let* $H_0 = \mathrm{Diag}(n,-1,\dots,-1)$ *and* $H = \mathrm{Diag}(1,-1,0,\dots,0)$ *in* $\mathfrak{sl}(n+1,\mathbb{C})$. *Then the orbit of* H_0 *in* $\mathfrak{sl}(n+1,\mathbb{C})$ *compactifies holomorphically and symplectically to a trivial product.*

Proof. For the examples we considered here, [GGS2] showed that $\mathcal{O}(H_0)$ can be embedded differentiably into $\mathbb{P}^n \times \mathbb{P}^{n*}$. As an outcome of our computations, we verify that the compactifications are also holomorphically and symplectically isomorphic to $\mathbb{P}^n \times \mathbb{P}^{n*}$. In fact, our package produces a compactification of the orbit embedded into $\mathbb{P}^{(n+1)^2-1}$ and the diamond shows that the compactified orbit has the topological type of a $\mathbb{P}^n$ bundle over $\mathbb{P}^n$, implying the bundle is trivial. □

Corollary 2. *An extension of the potential* f_H *to the compactification* $\mathbb{P}^n \times \mathbb{P}^{n*}$ *cannot be of Morse type; that is, it must have degenerate singularities.*

Proof. Our potential has singularities at $wH_0, w \in \mathscr{W}$. Now observe that the Hodge diamond of our compactified regular fibres have all zeroes in the middle row, hence any extension of the fibration to the compactification will have no vanishing cycles. However, the existence of a Lefschetz fibration with singularities and without vanishing cycles is precluded by the fundamental theorem of Picard–Lefschetz theory. □

6. Singular compactifications of $\mathfrak{sl}(3,\mathbb{C})$ orbits

We show that the topology of the compactified regular fibre for f_H can change drastically according to the choice of homogenisation of the ideal cutting out the orbit as an affine variety. The compactifications obtained in this section turn out to be singular.

6.1. **A fibration with 4 critical values.** In $\mathfrak{sl}(3,\mathbb{C})$ we take

$$H = H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is regular since it has 3 distinct eigenvalues. Then $X = \mathcal{O}(H_0)$ is the set of matrices in $\mathfrak{sl}(3,\mathbb{C})$ with eigenvalues $1, 0, -1$. This set forms a submanifold of real dimension 6 (a complex threefold).

In this case $\mathscr{W} \simeq S_3$, the permutation group in 3 elements, and acts via conjugation by permutation matrices. Therefore, the potential $f_H = x_1 - x_2$ has 6 singularities; namely, the 6 diagonal matrices with diagonal entries $1, 0, -1$. The four singular values of f_H are $\pm 1, \pm 2$. Thus, 0 is a regular value for f_H. Let $A \in \mathfrak{sl}(3,\mathbb{C})$ be a general element written as in (2), and let $p = \det(A)$, $q = \det(A - \mathrm{id})$. The ideals $\langle p, q\rangle$ and $\langle p-q, q\rangle$ are clearly identical and either of them defines the orbit though H_0 as an affine variety in $\mathfrak{sl}(3,\mathbb{C})$. Now

$$I = \langle p, q, f_H\rangle \qquad J = \langle p, p-q, f_H\rangle$$

are two identical ideals cutting out the regular fibre X_0 over 0. Let I_{hom} and J_{hom} be the respective homogenisations and notice that $I_{\mathrm{hom}} \neq J_{\mathrm{hom}}$, so that they define distinct projective varieties, and thus two distinct compactifications

$$\overline{X}_0^I = \mathrm{Proj}(\mathbb{C}[x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3, t]/I_{\mathrm{hom}}) \quad \text{and}$$
$$\overline{X}_0^J = \mathrm{Proj}(\mathbb{C}[x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3, t]/J_{\mathrm{hom}})$$

of X_0. Their Hodge diamonds are given in figure 3. Remark 4 explains the question marks.

Remark 4 (Computational pitfalls). Macaulay2 greatly facilitates calculations of Hodge numbers that are unfeasible by hand. However, the memory requirements rise steeply with the dimension of the variety – especially for the Hodge classes $h^{p,p}$. In fact, the unknown entries in our Hodge diamonds (marked with a '?') exhausted the 48GB of RAM of the computers of our collaborators at IACS without producing an answer.

6.1.1. *Expected Euler characteristic.* To reassure ourselves about the much larger values occurring for the diamond of $\overline{X}_0^I$ in comparison to $\overline{X}_0^J$, we perform the rather amusing calculation of the expected Euler characteristic of both varieties, which give out quite surprising numbers.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & & 16 & & ? & & ? & & 16 & & 0 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & & 1 & & ? & & ? & & 1 & & 0 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

FIGURE 3. The Hodge diamonds of two projectivisations $\overline{X}_0^I$ (left) and $\overline{X}_0^J$ (right) of the regular fibre corresponding to $H = H_0 = \mathrm{Diag}(1,-1,0)$.

Remark 5. Let $Y = Y_1 \cap \cdots \cap Y_r$ be a complete intersection. If Y is smooth, then the Euler characteristic of Y is uniquely determined by its cohomology class. However, for a singular variety this is no longer true, and the cohomological classes Y_i do not determine the topological Euler characteristic. They determine only what is called the expected Euler characteristic of Y (equal to the Fulton–Johnson class), see [Cy].

To calculate the expected Euler characteristic we use the following basic formulae from intersection theory. Let $X := \mathcal{V}(f_1, \ldots, f_k) \subset \mathbb{P}^{n+k}$ be a complete intersection with inclusion $i \colon X \to \mathbb{P}^{n+k}$. Define $\alpha := i^*\left(\mathrm{c}_1\left(\mathscr{O}_{\mathbb{P}^{n+k}}(1)\right)\right) \in H^2(X)$. Then

$$\int_X \alpha^{\wedge n} = d, \tag{3}$$

where $d = \prod_1^k d_i$ and $d_i = \deg f_i$. Moreover,

$$\mathrm{c}(X) = \frac{(1+\alpha)^{n+k+1}}{\prod_i (1 + d_i \alpha)} = 1 + \mathrm{c}_1(X) + \cdots + \mathrm{c}_n(X), \tag{4}$$

and the Euler characteristic is given by

$$\chi(X) = \int_X \mathrm{c}_n(X), \tag{5}$$

where $\mathrm{c}_i(X) \in H^{2i}(X)$ is the i-th Chern class.

Example 3. We first illustrate the formula with two elementary cases.

For a conic C in $\mathbb{P}^2$, expression 4 produces $(1+\alpha)^3/(1+2\alpha)$, whose expansion at zero is $1 + \alpha + \alpha^2 + o(\alpha^3)$. Here, $\int \alpha = 2$ and we get $\chi(C) = 2$, which was to be expected since the conic is topologically isomorphic to $\mathbb{P}^1$.

For the quartic Q in $\mathbb{P}^3$, expression 4 gives $(1+\alpha)^4/(1+4\alpha)$, whose expansion at zero is $1 + 6\alpha^2 + o(\alpha^3)$. Here, $\int \alpha^2 = 4$ and so $\chi(Q) = 6 \times 4 = 24$, which was to be expected since the quartic is a $K3$ surface, whose Hodge diamond is well known to be

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

Now let us return to our two projectivisations $\overline{X}_0^I$ and $\overline{X}_0^J$. For the ideal I_{hom} we have degrees $d_1 = d_2 = 3$ and $d_3 = 1$. The orbit was embedded in $\mathbb{P}^8$. So expression 4 gives

$$c\left(\overline{X}_0^I\right) = \frac{(1+\alpha)^9}{(1+3\alpha)(1+3\alpha)(1+\alpha)} = \frac{(1+\alpha)^8}{(1+3\alpha)^2}.$$

The Taylor series expansion around zero is given by $1+2\alpha+7\alpha^2-4\alpha^3+31\alpha^4-94\alpha^5+o(\alpha^6)$. Here $\int \alpha^5 = 9$ and we get the expected Euler characteristic to be

$$\chi\left(\overline{X}_0^I\right) = -94 \times 9 = -846.$$

On the other hand, for the ideal J_{hom} we have degrees $d_1 = 2$, $d_2 = 3$, and $d_3 = 1$. Expression 4 gives

$$c\left(\overline{X}_0^J\right) = \frac{(1+\alpha)^9}{(1+2\alpha)(1+3\alpha)(1+\alpha)} = \frac{(1+\alpha)^8}{(1+3\alpha)(1+2\alpha)}.$$

The Taylor series expansion around zero is $1+3\alpha+7\alpha^2+3\alpha^3+13\alpha^4-27\alpha^5+o(\alpha^6)$. In this case, $\int \alpha^5 = 6$ and we obtain

$$\chi\left(\overline{X}_0^J\right) = -27 \times 6 = -162.$$

The difference between $\chi\left(\overline{X}_0^J\right)$ and $\chi\left(\overline{X}_0^I\right)$ is another concrete topological difference between our two compactifications.

6.2. **A fibration with 6 critical values.** In $\mathfrak{sl}(3,\mathbb{C})$ we take

$$H_0 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

which is regular since it has 3 distinct eigenvalues. Then $\mathcal{O}(H_0)$ is the set of matrices in $\mathfrak{sl}(3,\mathbb{C})$ with eigenvalues $3,-1,-2$. We now choose

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

giving the potential $f_H(A) = x_1 - x_2$, with critical values $\pm 1, \pm 4, \pm 5$. This fibration is only mildly different from the previous one by the fact that 2 singular fibres contain 2 singularities each. The orbit is diffeomorphic to the one of subsection 6.1, The regular fibres are pairwise diffeomorphic.

As in 6.1, let $A \in \mathfrak{sl}(3,\mathbb{C})$, and $p = \det(A+\mathrm{id})$, $q = \det(A+2\,\mathrm{id})$. Once again, the ideals $\langle p,q\rangle$ and $\langle p-q,q\rangle$ are clearly equal and either of them defines the orbit though H_0 as an affine variety in $\mathfrak{sl}(3,\mathbb{C})$. The matrix A belongs to the regular fibre X_0 if in addition it satisfies $f_H = x_1 - x_2 = 0$. Now, let

$$I = \langle p,q,f_H\rangle \qquad J = \langle p,p-q,f_H\rangle$$

be two equal ideals cutting out the regular fibre X_0 through 0 and let I_{hom} and J_{hom} be the respective homogenisations. However, $I_{\text{hom}} \neq J_{\text{hom}}$, so they define distinct projective varieties. Performing the necessary computations, we obtain the same Hodge diamonds, and the same Euler characteristics as for the corresponding varieties of 6.1.

We then went further to check for the appearances of 16's and 1's in the Hodge diamonds of the singular fibres at 1 and indeed, they reappeared.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & ? & & 0 & & 0 & \\
0 & & 16 & & ? & & ? & & 16 & & 0 \\
 & 0 & & 0 & & ? & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & ? & & 0 & & 0 & \\
0 & & 1 & & ? & & ? & & 1 & & 0 \\
 & 0 & & 0 & & ? & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

FIGURE 4. The Hodge diamonds of two projectivisations of the singular fibre over 1 corresponding to $H_0 = \mathrm{Diag}(3,-2,-1)$, $H = \mathrm{Diag}(1,-1,0)$.

7. Open questions

We finish by posing the following open questions. How many compactifications can be obtained via homogenisation? Is there a preferred choice in the sense that it maintains the topology closest to the original variety? Given two compactifications with distinct numerical invariants, do there exist compactifications realising the intermediate values of the invariants?

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