Moduli Spaces - Instantons - Singularities

Mathematics

Problem 1: Topology of Moduli Spaces.

How does the topology of moduli $M_k(X)$ of vector bundles changes under birational transformations of the base X?

Theorem 1: Homology of Moduli Spaces.

Let Z be a compact surface containing a curve C with C^2 =-n and let X be obtained from Z by contracting the curve C. Then

$$H_2(\mathfrak{M}_k(Z), \mathfrak{M}_k(X)) \neq 0.$$

If n is even, then also

$$H_4(\mathfrak{M}_k(Z), \mathfrak{M}_k(X)) \neq 0.$$

Theorem 2: Atiyah Jones conjecture for rational surfaces.

For any rational surface X, and q < k/2

$$H_q(\mathfrak{M}_k(X)) = H_q(\mathfrak{M}_{k+1}(X))$$

Problem 2: Classification of singularities.

The classical invariants are not fine enough to distinguish inequivalent curve singularities.

Theorem 3: Instanton numbers of singularities.

Instanton numbers are finer than classical invariants.

Singularities					Instantons $j=3$		
polynomial	m	δ_P	μ	τ	w	h	charge
$x^3 - x^2y + y^3$	3	3	4	4	4	3	7
$x^3 - x^2y^2 + y^3$	3	3	4	4	5	3	8

Translation

Mathematics ➔ Physics

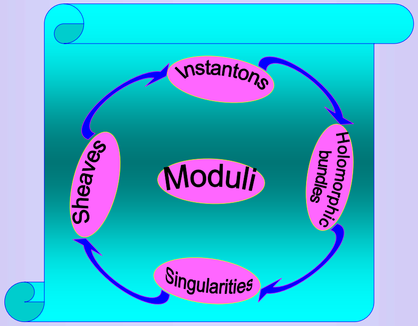
Kobayashi – Hitchin correspondence:
holomorphic bundles ⇔ Instantons

Moduli space of bundles with first Chern number 0 and second Chern number k ⇔ Moduli space of instantons of charge k.

Methods

Surgery on holomorphic bundles ⇨ Instanton "decay"

Construction of instantons ⇨ Construction of sheaves



Local charge ≡ local Euler characteristic
Instanton width ≡ co-length of direct image
Instanton height ≡ length of first derived image

Physics ➔ Mathematics

Contraction of spherical body with charge ⇨ Creation of singularity.

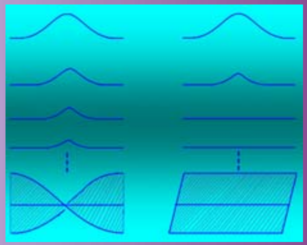
Instantons + contraction of spherical body ⇨ Existence of coherent sheaves with prescribed local Euler characteristic on singular varieties.

Simple representation of instantons near a spherical body by a pair (j,p) of an integer and a polynomial ⇨ Clear detection of obstruction to holomorphic surgery.

Physics

Problem 1': Instantons and topology of the base manifold.

How are instantons affected by the contraction of a spherical body ?



Obstructed partial decay   Total decay, unobstructed

The obstruction is determined by a compatibility condition between instanton numbers and the self-intersection of the sphere.

Theorem1': Obstructions to instanton decay.

The complexity of a neighborhood of the spherical body imposes an obstruction to instanton decay.

Theorem2': Stability of instanton moduli.

Instanton moduli have topological stability under increase of charge (homology in low degrees is maintained).

Theorem3': Instantons detect singularities.

Instanton numbers provide fine invariants for curve singularities.

Open questions

1. Stability of moduli in higher dimensions?
2. Relative homology of moduli of bundles on threefolds?
3. New invariants for hypersurface singularities? (the instanton methods used for curves readily generalizes to higher dimensions).
4. Physical interpretation of obstruction to instanton decay.

Dr. Elizabeth Gasparim, Department of Mathematics, New Mexico State University, gasparim@nmsu.edu