The Classification of Surfaces

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This is an expository paper which presents the holomorphic classification of rational complex surfaces from a simple and intuitive point of view, which is not found in the literature. Our approach is to compare this classification with the topological classification of real surfaces.

I. INTRODUCTION

Our aim here is to present the holomorphic classification of complex rational surfaces in a simple way. We base ourselves on the fact that the topological classification of real surfaces is very intuitive as one can see by following our pictures. One should keep in mind that a real surface is a two dimensional object while a rational surface is a four dimensional object. Hence, we can follow the classification of real surfaces by drawing pictures and use the intuition built in this case to understand the classification of rational surfaces, for which we are no longer able to draw pictures. When classifying real surfaces it is natural to take the following steps.

1. Basic surfaces : where we give the first elementary examples of real surfaces.
2. Two types of surfaces : where we show that real surfaces are naturally divided into two categories, namely orientable and non-orientable.
3. Constructing new surfaces out of basic ones : where we define the operation of connected sum and show how to build new examples of real surfaces.
4. The classification theorem : that says that all real surfaces can be obtained out of the basic ones by means of the connected sum operation.

We give an outline of the proof of the classification theorem for real surfaces.

Then we present the classification of rational surfaces following exactly the same four steps as above with the appropriate modifications.

1. Basic surfaces : where we give the first elementary examples of complex surfaces.
2. Two types of surfaces : where we show that complex surfaces are naturally divided into two categories, namely rational and irrational.
3. Constructing new surfaces out of basic ones : where we define the operation of blowing up and show how to build new examples of complex surfaces.
4. The classification theorem : that says that all rational surfaces can be obtained out of the basic ones by means of the blowing up operation.

We give an informal presentation and indicate references for detailed proofs. We only assume some elementary knowledge of smooth manifolds and some linear algebra. By a real surface we mean a smooth manifold of two real dimensions, which is connected and compact. By a complex surface we mean a complex smooth manifold of two complex dimensions (hence four real dimensions) which is connected and compact.

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II. THE TOPOLOGY OF REAL SURFACES

1. Basic surfaces. We begin with examples of real surfaces.

- the sphere S^2
- the torus T (which looks like a doughnut)
- the n-torus nT (which looks like a doughnut with n holes)

The orientable surfaces

fig. 1. The Sphere   fig. 2. The Torus   fig. 3. The 2-torus

- the real projective plane $\mathbf{RP}^2 \sim \frac{\mathbf{R}^3 - \{0\}}{z \sim \lambda z}$, for $\lambda \in \mathbf{R} - \{0\}$.
- the Klein bottle K, which can be viewed as a twisted torus, constructed by gluing the two ends of a cylinder with a twist.

A Non-orientable surface

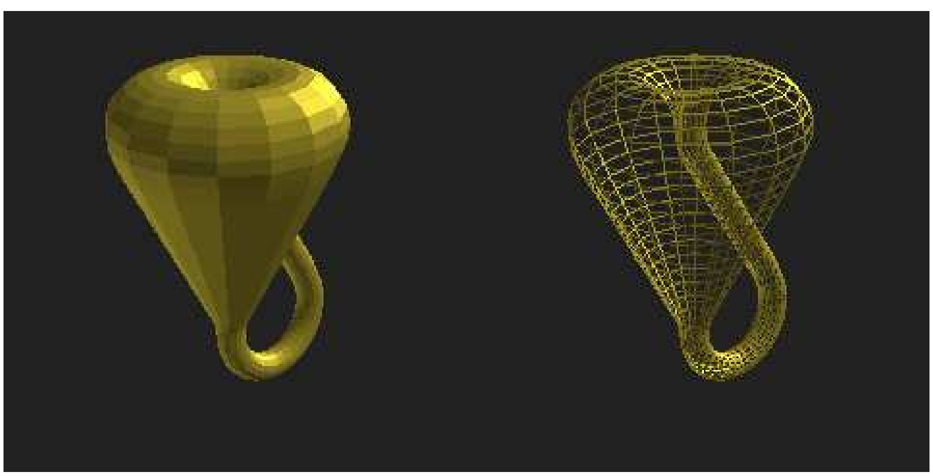

fig. 4. The Klein Bottle

2. Two types of surfaces.

There is a natural division in the classification of real surfaces, given by orientability. Orientable surfaces are intuitively the ones that have two sides, that is inside and outside, like S^2, T, and nT. Orientable real surfaces are the so called Riemann surfaces. The non-orientable surfaces are the ones containing a Möbius band, which therefore don't have outside and inside, like $\mathbf{RP}^2$ and K.

3. Constructing new surfaces out of the basic ones.

When one wants to study the topology of real surfaces, there is a basic operation, which allows us to construct new surfaces out of old ones. This operation is called connected sum (represented by #), and works as follows. Given two surfaces S_1 and S_2, cut out open discs D_1 in S_1 and D_2 in S_2 and then glue the two surfaces $S_1 - D_1$ and $S_2 - D_2$ by identifying the boundaries of D_1 and D_2. Here are some examples.

- $X \# S^2 = X$ for any real surface X, that is, the sphere works as an identity for the # operation
- $T \# T = 2T$, that is, the 2-torus is obtained by gluing together two tori (Intuitively, just put together two doughnuts one after the other and you get a 2-torus.)

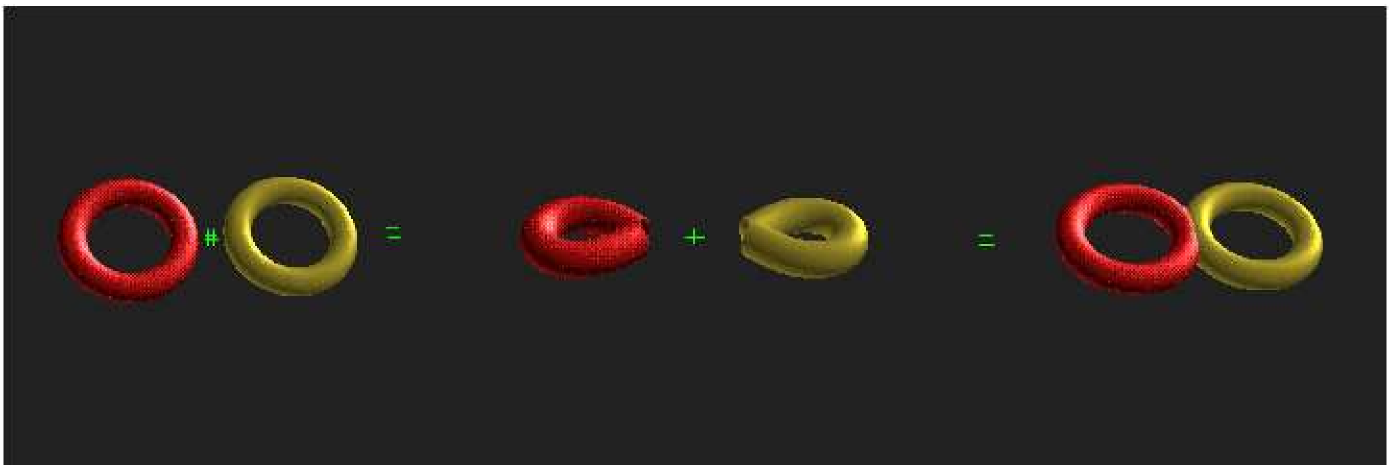

fig. 5. The Connected Sum of Two Tori

- $nT \# mT = (n+m)T$, which is a generalization of the previous example.
- $\mathbf{RP}^2 \# \mathbf{RP}^2 = K$. This case is not intuitively obvious, but is very easy to show by cuting and pasting. We give a proof of this in section III.

4. The classification theorem.

Once we have the basic surfaces and the operation for constructing new surfaces out of old ones, we can build all real surfaces. The classical structure theorem is the following (see [2]).

Theorem II.1 *Every compact connected real surface is obtained from either the sphere S^2 or the torus T or the real projective plane $\mathbf{RP}^2$ by connected sums.*

III. THE PROOF OF THE STRUCTURE THEOREM

In this section we outline the proof of the classification theorem for real surfaces. For a detailed proof see [2]. By the very definition of the connected sum operation, we know that performing connected sums on real surfaces, we always obtain a real surface. However, it is not obvious that all real surfaces are obtained by connected sums starting only with the sphere, the torus and the real projective plane. So in this section, we will give an idea of how to show this.

To prove this theorem one very useful tool is to construct surfaces by identifying sides of polygons. Here one should keep in mind that we want a topological classification, so that an object is considered the same as a continuous deformation of it which can be reversed. For example, from a topological point of view a disc and a square are considered the same topological object, since each can be continuously deformed to the other.

Here are some examples of representing real surfaces by polygons with sides identified. We will name our polygons according to their sides read counterclockwise.

The Polygon Representations

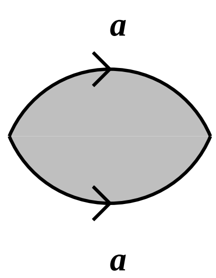

fig.6 The Sphere

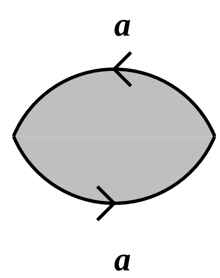

fig.7 The Real Projective Plane

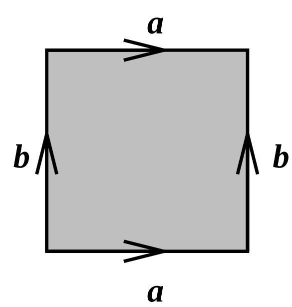

fig.8 The Torus

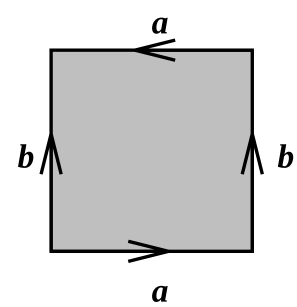

fig.9 The Klein Bottle

Now we also need to represent connected sums using this method of polygons. This is easily seen by an example.

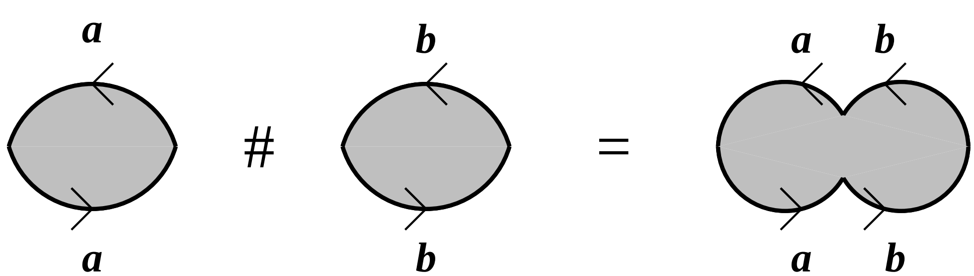

fig.10 Schematic representation of the sum of two real projective planes

We can represent any of our real surfaces by a polygon with sides identified. However this representation is not unique. To understand this look at the following sequences which shows that $RP^2 \# RP^2$ equals the Klein bottle. Let us simply continue from the previous sequence of pictures. We cut our polygon with sides *aabb* through a diagonal side c.

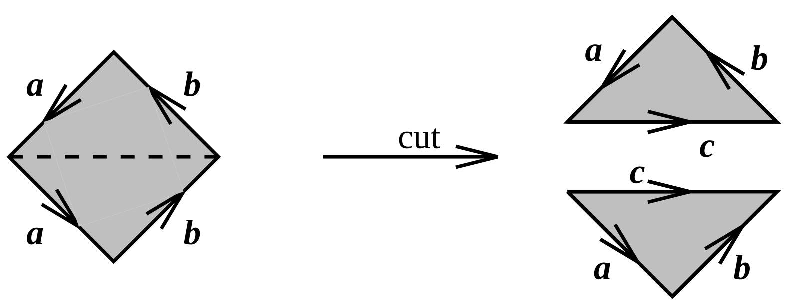

fig.11 The Cut

Then we glue the two triangles back together by their b sides to obtain the sequence of sides $aca^{-1}c$ which is the Klein bottle.

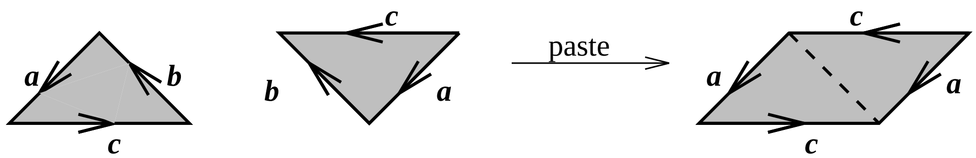

fig.12 The Paste

Now we are ready to understand the proof of the structure theorem of real surfaces. The proof goes as follows. Every real surface is obtained by identifying sides of a polygon. This is intuitively believable. The rigorous mathematical reason is that every surface has a triangulation.

Then to prove the theorem one wants to see that one can choose a representation which shows the correspondence with a connected sum of tori, spheres or real projective planes. First of all it is clear that each letter appears in two sides of the polygon since we want to obtain a closed surface. Hence for every surface we get a sequence of letters where each letter appears twice. We put the exponent -1 when a letter appears in a side pointing in the clockwise direction. Then we use the operations of cutting and pasting as we did in the above example to get the polygon to a nice form. One shows that cutting and pastings can be done to get the sequence of letters to be always of one of the types aa, aa^{-1} or $aba^{-1}b^{-1}$ following each other any number of times. For example we can have $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_nb_na_n^{-1}b_n^{-1}$ which is a connected sum of n tori. Mathematically one gets a group generated by expressions of the form aa, aa^{-1} and $aba^{-1}b^{-1}$. Each geometric property of a surface gets translated into a property of the group. For example, the fact that the sphere is an identity element for connected sums gets translated into the fact that $xaa^{-1} = x$ in the group language. The best way to understand this result is to draw polygons on pieces of papers and perform cutting and pasting until one gets to the form of connected sums of the basic elements.

Summing up, given a surface, one represents it by a polygon with sides to be identified. Then by cutting and pasting one can always get to a simple representation of the surface as a polygon containing only sequences of sides of the form aa, aa^{-1} and $aba^{-1}b^{-1}$. These are exactly the expressions that correspond to RP^2, S^2 and T.

IV. THE STRUCTURE OF RATIONAL SURFACES

Here we imitate step by step the classification of real surfaces presented in section II. We begin with some examples.

1. Basic surfaces. Here we give examples of complex surfaces.

- the complex projective plane $\mathbf{CP}^2$
- the product of two projective lines $\mathbf{CP}^1 \times \mathbf{CP}^1$
- The Hirzebruch surfaces S_n.

For the first two examples, recall that $\mathbf{CP}^n := \frac{\mathbf{C}^{n+1}-\{0\}}{z\sim\lambda z}$, for $\lambda \in \mathbf{C} - \{0\}$. To define the Hirzebruch surfaces, we need the concept of a vector bundle. Intuitively, a rank n vector bundle over a manifold M is given by attaching to each point of the manifold a rank n vector space F, which is called the fiber. In other words a vector bundle over M is a family of vector spaces parametrized by the points of M. One very basic property of vector bundles that one should keep in mind is called local triviality. This means that when we look at a small open set U in the manifold M, then the vector bundle over U is isomorphic to a trivial product $U \times F$. Here we only need vector bundles over $\mathbf{CP}^1$, and we will only define these. For more on vector bundles see [4].

First of all we choose charts for the complex projective line as follows: $\mathbf{CP}^1 = U \cup V$, where $U \simeq V \simeq \mathbf{C}$ with intersection $U \cap V \simeq \mathbf{C} - \{0\}$, and transition function $z \to z^{-1}$. Topologically $\mathbf{CP}^1$ is just the sphere S^2 and one can look at the chart U as covering the sphere minus the north pole, while V covers the sphere minus the south pole.

Now we define the rank one complex vector bundle $\mathcal{O}(n)$ over $\mathbf{CP}^1$ (rank one complex means that the fibers will be copies of $\mathbf{C}$). To define this bundle we start with two trivial products $U \times \mathbf{C}$ and $V \times \mathbf{C}$. Then we need to give a transition function that tells us how the fibers get put together in the overlap of the charts. For each n we give the function z^{-n}, which means that we identify these fibers by the vector space isomorphism taking u to $z^{-n}u$. Formally we define $\mathcal{O}(n)$ by charts $U' = U \times \mathbf{C} \simeq \mathbf{C}^2$, $V' = V \times \mathbf{C} \simeq \mathbf{C}^2$, with intersection $U' \cap V' \simeq \mathbf{C} - \{0\} \times \mathbf{C}$, and transition function $(z, u) \to (z^{-1}, z^{-n}u)$.

Now we construct rank two vector bundles over $\mathbf{CP}^1$ using the operations we know for vector spaces coming from basic linear algebra. Recall that we have an operation of direct sum for vector spaces, which for vector spaces V_1 of rank r_1 and V_2 of rank r_2 associates the vector space $V_1 \oplus V_2$ of rank $r_1 + r_2$. Now starting with two vector bundles E_1 and E_2 over a manifold M, we construct a new bundle over M called the Whitney sum $E_1 \oplus E_2$, whose fibers are just the vector space sum of the fibers of E_1 and E_2. That is if E_1 has a fiber V_1 at the point p and E_2 has a fiber V_2 at the point p then $E_1 \oplus E_2$ has fiber $V_1 \oplus V_2$ at the point p. For example, taking the Whitney sum of the bundles $\mathcal{O}(m)$ and $\mathcal{O}(n)$ will give us a rank two bundle over $\mathbf{CP}^1$ denoted by $\mathcal{O}(m) \oplus \mathcal{O}(n)$.

Now, just as we constructed $\mathbf{RP}^2$ by projectivizing $\mathbf{R}^3$ and $\mathbf{CP}^2$ by projectivizing $\mathbf{C}^3$, it is possible to projectivize any vector bundle E, thus obtaining a bundle $P(E)$ whose fibers are projective spaces. Also here the idea is to perform the projectivization at each of the fibers. If we start with the rank two bundles $\mathcal{O}(n) \oplus \mathcal{O}(0)$ then projectivizing we transform the fibers into projective lines $\mathbf{CP}^1$. Recall that these are all bundles over $\mathbf{CP}^1$. Hence what we obtain is actually a family of $\mathbf{CP}^1$'s parametrized by points of another copy of $\mathbf{CP}^1$. The reader should intuitively think of the number n as a particular way of twisting such a family n times when we go around the base.
We are now ready to define the n-th Hirzebruch surface:

$$S_n = P(\mathcal{O}(n) \oplus \mathcal{O}(0)).$$

Each Hirzebruch surface S_n is a complex surface. Hence now we have an infinite set of complex surfaces, one for each n. We remark also that the first Hirzebruch surface S_0 is simply the product $\mathbf{CP}^1 \times \mathbf{CP}^1$, intuitively in this case we have a family of $\mathbf{CP}^1$'s parameterized by a $\mathbf{CP}^1$ with no twisting hence a product.

2. Two types of surfaces.

Just as real surfaces were naturally divided into two categories, also complex surfaces will be divided into two categories. However orientability does not make a division, because all complex surfaces are orientable. There is a natural division in the classification of complex surfaces is given by rationality. Let us observe that all complex surfaces we mention here can be thought of as lying inside some complex projective space. In a projective space it always make sense to write down quotients of homogeneous polynomials of the same degree. One then defines a rational map to be a map that is locally given (in each coordinate) by quotients of polynomials. We remark a very special property of rational maps that is the fact that a rational map need not be defined over every point of its domain. Clearly when we define a map by quotients of polynomials, then the map is not defined where the denominator vanishes. In fact a rational map needs only to be defined over a dense open subset of its domain and two rational maps are considered equal if they coincide on a dense open set.

Rational surfaces are intuitively the ones that are somehow similar to $\mathbf{CP}^2$. By definition, a complex surface S is rational if there is a rational map $\Phi : S \to \mathbf{CP}^2$, whose inverse is also rational. Where two rational maps are called inverse to each other if their composition is the identity defined over a dense open set. Obviously $\mathbf{CP}^2$ itself is rational. Also the Hirzebruch surfaces are rational surfaces and these are in fact the simplest examples of rational surfaces.

3. Constructing new surfaces out of basic ones.

When one wants to study all rational surfaces, there is a basic operation, which allows us to construct new surfaces from old ones. This operation is called blow-up. When we start with a complex surface S and blow up a point in S, the result is another complex surface $\widetilde{S}$ which is different from S. In fact topologically, the new surface $\widetilde{S}$ is obtained from S by performing a connected sum of S together with a copy of $\mathbf{CP}^2$ with the reversed orientation. The orientation problem is a technical detail that we will explain after we define the blow-up. But right now it is important to see that when we do a blow-up, then from a topological point of view we are doing exactly the same kind of operation that we did for real surfaces in section II. Hence some of the geometric intuition we got from the pictures can be carried over to this section. However, we remark also that the blow-up operation is an analytic procedure in the sense that it takes us from a complex analytic manifold into another complex analytic manifold. That means that the blow-up operation is much more powerful then simply performing connected sums in a topological way as we did for real surfaces. We could phrase this as *the blow-up is a complex analytic way of performing connected sums.*

The blow-up operation is defined locally, in the sense that it is done in a coordinate chart. That is, to blow-up a point p in a surface S, we take a coordinate chart around p, taking p to $0 \in \mathbf{C}^2$. Then we blow up the origin in $\mathbf{C}^2$ and construct the surface $\widetilde{S}$, which is isomorphic to S everywhere except at the point p, which gets replaced by a

$\mathbf{CP}^1$. Intuitively one picks a point and blows it up (in the usual sense of the word, i.e. to explode) to a projective line. The result is that the lines passing through p become disjoint. The next following picture represents the blow-up of $\mathbf{R}^2$ at the origin. Note that after the blowup we get a mobius band as the top and bottom edges of the strip are identified after giving a twist. A good way to understand this picture is to think of the blow-up as placing a line L through origin at the height θ where θ is the angle of L. Then the line corresponding to 2π is to be identified to the line corresponding to zero, since they coincide in $\mathbf{R}^2$.

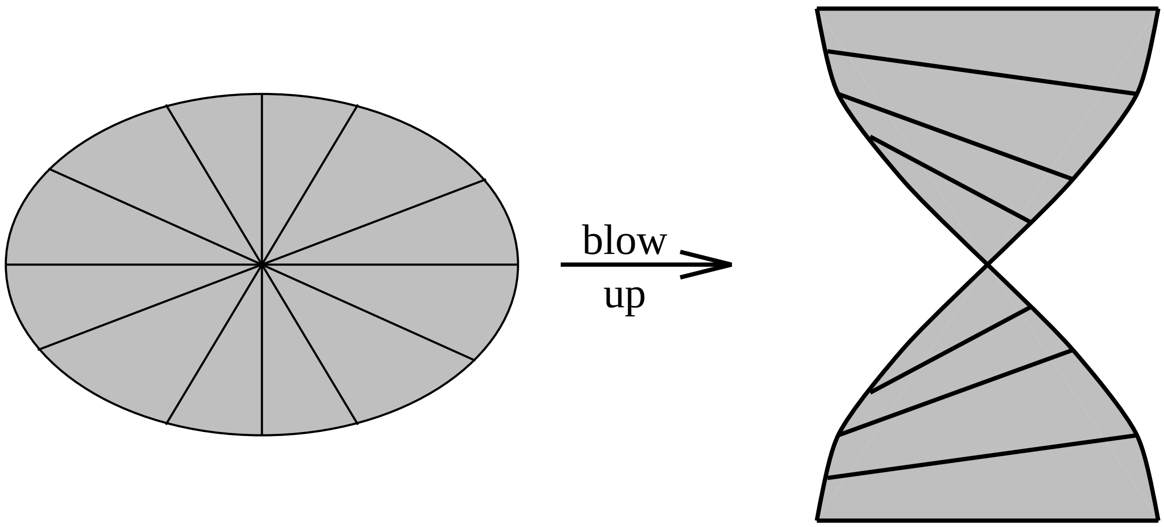

fig.13 The blow up of a point [2]

Let us now write the formal definition of the blow up of $\mathbf{C}^2$ at the origin, denoted by $\widetilde{\mathbf{C}^2}$, which is:

$$\widetilde{\mathbf{C}^2} = \{(z, l) \in \mathbf{C}^2 \times \mathbf{CP}^1 / \, z \in l\}.$$

Notice that if $z \in \mathbf{C}$ is nonzero, then z determines a unique line l namely the line through the origin in $\mathbf{C}^2$ passing by z. Hence, over $z \neq 0$ $\widetilde{\mathbf{C}^2}$ is isomorphic to $\mathbf{C}^2$, which is what we should expect since points outside of the origin are not blown-up. However, for $z = 0$ we have that $(0, l) \in \widetilde{\mathbf{C}^2}$ for every l. This gives a copy of $\mathbf{CP}^1$ which lies over the origin, called the exceptional divisor. This exceptional divisor "separates" the lines through the origin turning them into disjoint lines. One very nice exercise for the reader would be to show that $\widetilde{\mathbf{C}^2}$ is the line bundle $\mathcal{O}(-1)$ over $\mathbf{CP}^1$. But for a first approach the reader may just take this as a result.

A copy of $\mathbf{CP}^1$ inside a surface is called a line. The exceptional divisor is a special kind of line. For instance it is different from any line contained in $\mathbf{CP}^2$. We now assign numbers to lines according to the form of their neighborhoods, this number is know as the self-intersection number of the line. A line $\mathbf{CP}^1$ inside $\mathbf{CP}^2$ will be our standard line. To it we assign the number 1. To the exceptional divisor we will assign the number -1 (the technical reason for this is the exercise we just left to the reader).

For a line which has a neighborhood isomorphic to $\mathcal{O}(n)$ we assign the number n. These numbers are the analogous to the directions for the arrows we had when dealing with real surfaces. Only in that case only two direction were possible, so an arrow was enough. Now we have "twists" and we label the lines according to the twists in their neighborhoods. A negative sign can be thought of as meaning twist in negative direction. It is this negative sign that accounts for the reversed orientation when we talked about the topology of the blow-up viewed as a connected sum.

An interesting example is to start with $\mathbf{CP}^2$ and blow up two points p and q, and then we blow down the line passing by p and q. Then, the resulting surface is $\mathbf{CP}^1 \times \mathbf{CP}^1$. This is a lot more technical, we refer the reader to [1] for a proof.

Not only the blow-up is topologically what we would expect for a generalization of the operation we had for real surfaces and even more, an analogous structure theorem holds.

4. The classification theorem.

Once we have the basic surfaces and the operation for constructing new surfaces out of old ones, we can build all rational surfaces. The classical structure theorem is the following (see [1]).

Theorem IV.1 *Every compact connected rational surface is obtained from either the complex projective plane* $\mathbf{CP}^2$ *or from a Hirzebruch surface* S_n *by blowing up points.*

A complete proof of this theorem would require that we develop quite a bit of theory. Hence we give only some informal ideas. We start with a rational surface and look for special lines inside the surface. Then there are two possibilities either we have lines with associated number -1 or we do not have any such lines. Suppose S is a surface containing a -1 type line l. Then we can eliminate this line l by performing what is called a blow-down, that is exactly the inverse operation of a blow-up. The blow-down of l contracts the line into a point p giving a new surface S' (so that the blow-up of S' at p equals S). Then if S' has no more -1 type lines we are done. Otherwise we perform another blow-down on S'. It is true that after finitely many blowing downs we arrive at a surface containing no more -1 type lines. Then there are two possibilities, either the new surface has only type 1 lines and in this case it is $\mathbf{CP}^2$ or it has some line of type $-n$, in which case it is a Hirzebruch surface S_n.

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