

**Moduli of bundles on
local surfaces and threefolds**

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Abstract

In this thesis we study the moduli of holomorphic vector bundles over a non-compact complex space X , which will mainly be of dimension 2 or 3 and which contains a distinguished line $\ell \subset X$. We will consider the situation in which X is the total space of a holomorphic vector bundle on $\mathbb{CP}^1$ and ℓ is the zero section.

While the treatment of the problem in this full generality requires the study of complex analytic spaces, it soon turns out that a large part of it reduces to algebraic geometry. In particular, we prove that in certain cases holomorphic vector bundles on X are algebraic.

A key ingredient in the description of the moduli are numerical invariants that we associate to each holomorphic vector bundle. Moreover, these invariants provide a local version of the second Chern class. We obtain sharp bounds and existence results for these numbers. Furthermore, we find a new stability condition which is expressed in terms of these numbers and show that the space of stable bundles forms a smooth, quasi-projective variety.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Thomas Köppe)

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Chapter 1

Preliminaries

1.1 Introduction

The aim of this thesis is to add to the understanding of the moduli of holomorphic vector bundles on non-compact complex spaces. The cases we consider are complex surfaces and threefolds which are the total spaces of bundles over $\mathbb{CP}^1$. Both these cases are not only interesting in geometry, but also in mathematical physics. Indeed, there is an extensive theory relating holomorphic vector bundles on smooth complex surfaces to instantons on the underlying real manifold, provided by Kobayashi-Hitchin correspondence. The three-dimensional case, on the other hand, is interesting in string theory, in which holomorphic bundles, or more generally coherent sheaves, describe string boundary conditions (so-called D -branes). A description of the moduli of such bundles is therefore important for any type of problem that requires integration over “all branes”, which is a staple of mathematical physics.

Several results of this thesis have been published in joint work with my supervisor E. Gasparim and with physicist P. Majumdar and E. Ballico. Some results will only be cited, while the proofs of others are repeated here. By and large, lots of the results on complex surfaces (Chapter 2) have been published, while the material in Chapter 3 on threefolds is new.

The Kobayashi-Hitchin correspondence between irreducible $SU(2)$ -instantons and stable holomorphic vector bundles of rank 2 was proved for compact Kähler surfaces by Uhlenbeck and Yau, for $\mathbb{C}^2$ by Donaldson and for $\widehat{\mathbb{C}^2}$, the blow-up of the plane in the origin, by King. The result was extended to the non-compact spaces Z_k described below in [GKM08], where $Z_1 = \widehat{\mathbb{C}^2}$.

When passing from complex projective geometry to non-compact spaces, one immediately faces the complication that there exist holomorphic objects that are not algebraic. We will briefly review the basic definitions of the categories of complex schemes and analytic spaces, before demonstrating that the class of non-compact spaces of the form $Z_k := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ for $k > 0$ satisfies GAGA-type properties, as does the space $W_1 := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. Armed with this knowledge, we are able to present an explicit description of holomorphic vector bundles on Z_k and W_1 and to attempt a first guess at how to parametrise their moduli.

The main part of this thesis consists of the construction of several holomorphic numerical invariants of vector bundles. These techniques are applicable both in the case of surfaces and of threefolds, and indeed they generalise to spaces of higher dimensions and bundles of higher rank. The crucial condition on the base space is that it contains a contractible line $\ell \cong \mathbb{P}^1$. If Z denotes any such space in question, we write $\pi: Z \rightarrow X$ for the contraction of ℓ (so for example we have $\pi: Z_1 = \widehat{\mathbb{C}^2} \rightarrow \mathbb{C}^2$). If E denotes a holomorphic vector bundle

on Z , then two of the numerical invariants of E are obtained from the cohomology of the direct image $R\pi_*E$. We construct further invariants (which do not necessarily require the space to contain a contractible line, merely a compact 1-cycle) from the cohomology of the endomorphism sheaf $\mathcal{E}nd E$.

The computation of these numbers proceeds by iteration over infinitesimal neighbourhoods. We will discuss the distinction between the algebraic and the analytic category and conclude that we obtain the same results by performing the computations in either category. Finally, I developed a set of computer algorithms for the computation of the invariants, using the great open-source computer algebra system *Macaulay 2* by Grayson and Stillman. While a detailed description of the implementation is left to a separate publication, the results of these automated computations have been used in several results.

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1.2 Analytic and algebraic geometry

The objects of our study lie at the confluence of different fields of mathematics, namely topology, differential geometry, analysis and algebra. To study the geometry of a space X , we will need to know its topology and its differential structure, so the notion of smooth manifolds and vector bundles enters, but this is not quite enough. To fully express the subtleties that arise, we need the notion of coherent sheaves over schemes and analytic spaces, or even over formal schemes, formal spaces and stacks.

To begin, we will introduce two related notions of *analytic spaces* and *schemes*. To this end, we first define several basic algebraic notions.

1.2.1 Basic definitions

We assume familiarity with basic notions of group and ring theory. In particular, every Abelian group is a $\mathbb{Z}$ -module and every ring with unit is a $\mathbb{Z}$ -algebra, so it suffices to study modules and algebras.

Algebra. We will write $\mathbb{C}\{x_1, \dots, x_n\}$ for the $\mathbb{C}$ -algebra of power series in n variables that converge on a neighbourhood of $0 \in \mathbb{C}$. If $\mathbb{k}$ is any field (or indeed commutative ring with unit), we will write $\mathbb{k}[x_1, \dots, x_n]$ for the $\mathbb{k}$ -algebra of polynomials in n variables. Clearly $\mathbb{C}[x_1, \dots, x_n] \subsetneq \mathbb{C}\{x_1, \dots, x_n\}$.

We will also reserve the notation $R, S, \dots$ for commutative rings with unit. We denote by $\mathcal{M}\text{od}_R$ the (Abelian) category of R -modules and by mod_R the full subcategory of finitely generated R -modules. If $\mathcal{A}$ is an Abelian category, we write $C(\mathcal{A})$ for the (Abelian) category of cochain complexes, $K(\mathcal{A})$ for the (Abelian) category of cochain complexes up to cochain homotopy, and $D(\mathcal{A})$ for the derived category of $\mathcal{A}$. We also write $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ for the full subcategories of bounded (respectively above, below and both) complexes, and for $C(\mathcal{A})$ and $D(\mathcal{A})$ similarly. Note that $K(\mathcal{A})$ and $D(\mathcal{A})$ are naturally triangulated. We write $C_{\text{mod}_R}(\mathcal{M}\text{od}_R)$ for the category of cochain complexes in $\mathcal{M}\text{od}_R$ whose cohomologies lie in mod_R , and similarly for K and D .

Sheaves. If X is any topological space, there is a category $\mathcal{O}\text{pen}_X$ whose objects are the open sets of X and whose morphisms are the inclusions. A (*set-valued*) *presheaf* on X is a functor $\mathcal{F} \in [\mathcal{O}\text{pen}_X^{\text{op}}, \mathcal{S}\text{et}]$, where $\mathcal{S}\text{et}$ is the category of sets, $\mathcal{C}^{\text{op}}$ denotes the opposite category of a category $\mathcal{C}$ and $[\mathcal{C}, \mathcal{D}]$ denotes the category of functors from a category $\mathcal{C}$ to a category $\mathcal{D}$, whose morphisms are natural transformations. A *presheaf of Abelian groups* takes values in the concrete category of Abelian groups, and a *presheaf of rings, modules, algebras* etc. takes values in the respective concrete subcategories. A presheaf is a *sheaf* if the *gluing axiom* holds: For any two open subsets $U, V \subseteq X$, if there exist $s_1 \in \mathcal{F}(U)$ and $s_2 \in \mathcal{F}(V)$ such that $s|_{U \cap V} = t|_{V \cap U}$, then there exists $t \in \mathcal{F}(U \cup V)$ such that $t|_U = s_1$ and $t|_V = s_2$. For every point $x \in X$, the *stalk at x* of a presheaf $\mathcal{F}$ is

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U),$$

i.e. elements of $\mathcal{F}_x$ are represented by pairs (V, s_V) where V is open and contains x , $s_V \in \mathcal{F}(V)$, and $(V', s_{V'})$ is equivalent to (V, s_V) if and only if $x \in V \cap V'$ and $s_V|_{V \cap V'} = s_{V'}|_{V' \cap V}$.

1.2.2 Geometric spaces

Definition 1.1. A *ringed space* is a pair $(X, \mathcal{A})$, where X topological space and $\mathcal{A}$ is a sheaf of commutative rings with unit on X . A *locally ringed space* is a ringed space $(X, \mathcal{A})$ where each stalk $\mathcal{A}_x$ is a *local ring*, i.e. a ring with a unique maximal ideal, which we denote by $\mathfrak{m}_x$. We will also write $\mathcal{A} =: \mathcal{O}_X$ and call $\mathcal{O}_X$ the *structure sheaf* of X .

Definition 1.2. If $(X, \mathcal{A})$ is a ringed space, we say that a sheaf $\mathcal{F}$ on X is an $\mathcal{A}$ -*module* if each $\mathcal{F}(U)$ is an $\mathcal{A}(U)$ -module and the induced maps are module homomorphisms. We call an $\mathcal{A}$ -module $\mathcal{F}$ *locally free* if $\mathcal{F}(U)$ is a free $\mathcal{A}(U)$ -module for all open sets $U \subseteq X$; equivalently if all stalks $\mathcal{F}_x$ are free $\mathcal{A}_x$ -modules.

Analytic spaces. Note that the algebra $\mathbb{C}\{x_1, \dots, x_n\}$ is a local algebra, i.e. it has a maximal ideal (namely the ideal of power series without constant term). We call an algebra $\mathcal{A}$ a $\mathbb{C}$ -*analytic algebra* if it is isomorphic to the quotient of $\mathbb{C}\{x_1, \dots, x_n\}$ by some finitely generated ideal for some n . (A similar notion exists for real-analytic algebras.)

Definition 1.3. A $\mathbb{C}$ -*analytic space* is a locally ringed space $(X, \mathcal{O}_X)$ where each stalk $\mathcal{O}_{X,x}$ is a $\mathbb{C}$ -analytic algebra. The structure sheaf $\mathcal{O}_X$ is the sheaf of germs of holomorphic functions, whose stalks consist of power series that converge on some neighbourhood.

Remark 1.4. Every analytic space looks locally like $\{f_1 = \dots = f_r = 0\} \subset \mathbb{C}^n$, where the f_i are holomorphic functions on $\mathbb{C}^n$, and the corresponding analytic algebra is just $\mathbb{C}\{x_1, \dots, x_n\}/(f_1, \dots, f_r)$. In fact, the *anti-equivalence principle* says precisely that *germs of analytic spaces* correspond

precisely to analytic algebras. We call the induced topology on X the *Euclidean* or *analytic* topology.

Schemes. (See standard textbooks like [Har77].) To every commutative ring with unit R we can associate a locally ringed space $(\text{Spec}(R) = X, \mathcal{A})$ such that $\mathcal{A}(X) = R$; this space $\text{Spec}(R)$ is called an *affine scheme*. Its points are the prime ideals of R , and its *closed* points are the maximal ideals. The topology coming from the Spec-construction is the *Zariski topology*, in which closed sets are precisely the zero locus of polynomials.

A general *scheme* can be covered by open sets that are affine schemes. If R is the quotient of $\mathbb{C}[x_1, \dots, x_n]$ by a finitely generated ideal, we say that $\text{Spec}(R)$ is an affine scheme over $\mathbb{C}$ (and similarly for general schemes). Note that such a scheme over $\mathbb{C}$ is a locally ringed space whose structure sheaf is the *sheaf of regular functions*, whose stalks are germs of *polynomial functions*. Equivalently, such a scheme is locally the zero locus $\{f_1 = \dots = f_r = 0\} \subset \mathbb{C}^n$ of *polynomials*.

Analytification. Since a polynomial ring over $\mathbb{R}$ or $\mathbb{C}$ is contained in the ring of convergent power series and the latter is a module over the former, every real or complex scheme defines uniquely a real- or complex-analytic space, which we may call the *analytification* of the scheme.

Formal spaces and schemes. If $(X, \mathcal{O}_X)$ is a complex space or scheme and $\mathcal{I} \subset \mathcal{O}_X$ a sheaf of ideals defining a subspace $A \subseteq X$, then the locally ringed space

$$A^{(m)} := (A, \mathcal{O}_X / \mathcal{I}^{m+1}|_A)$$

is called the m^{th} *infinitesimal neighbourhood* of A in X ; it is itself respectively a complex space or scheme. Moreover, for varying m these neighbourhoods form an inverse system $\dots \rightarrow A^{(m)} \rightarrow A^{(m-1)} \rightarrow \dots \rightarrow A^{(0)} = A$. We call the inverse limit of this system the *formal completion of X along A* , written $\hat{A}$. Note that when $\mathcal{I} = 0$, then $A = X$ and $\hat{X} = X$. We will colloquially call $\hat{A}$ the *formal neighbourhood* of A .

For example, the formal completion of the origin in affine n -space is given by the limit

$$\varprojlim_m \mathbb{C}[x_1, \dots, x_n] / (x_1, \dots, x_n)^{m+1} = \mathbb{C}[[x_1, \dots, x_n]],$$

given by the ring of *formal* power series in the n variables. In analogy with the Spec-construction $\mathbb{C}^n = \text{Spec} \mathbb{C}[x_1, \dots, x_n]$, we also speak of a *formal spectrum* and write $\hat{0} = \text{Spf} \mathbb{C}[[x_1, \dots, x_n]]$. Finally, a *formal complex space* or a *formal scheme* is a space that is covered by open sets that are formal spectra. In other words, formal spaces or schemes look locally like the formal completion of a space along a subspace. By virtue of our earlier remark, every complex space is also a formal complex space, and likewise for schemes.

Note as an aside that the notion of formal completion is always available in Algebraic Geometry, over any ground field, while the notion of analyticity and convergence exist mainly over $\mathbb{R}$ or $\mathbb{C}$.

1.3 Moduli and deformation theory

1.3.1 Informal introduction

The *moduli problem* is, in a very general sense, the question whether there exists an object $\mathfrak{M}$, the *moduli*, that parametrises all objects of a certain type – for example, all vector bundles

over a scheme. In that case, each point of $\mathfrak{M}$ is one such object. This situation is particularly interesting if $\mathfrak{M}$ itself has geometric structure: In good cases, when parametrising algebraic objects over a scheme, $\mathfrak{M}$ might be a scheme itself, or a more general object like an *algebraic space* or a *stack*, the notions of which were invented precisely to describe the solutions of moduli problems.

Suppose now that $\mathfrak{M}$ parametrises vector bundles over a fixed base space X up to isomorphisms, so that we may write $[E] \in \mathfrak{M}$ for the point that parametrises all bundles isomorphic to $E \rightarrow X$. If $\mathfrak{M}$ is smooth at $[E]$, the tangent space $T_{[E]}\mathfrak{M}$ measures infinitesimal first-order deformations of E . Intersection theory tells us what $T_{[E]}\mathfrak{M}$ is (if $\mathfrak{M}$ has a perfect obstruction theory), and in the case of vector bundles over a projective scheme X it will be $H^1(X; \mathcal{E}nd E)$. The upshot is that the dimension of this cohomology group is the dimension of the component of the moduli containing $[E]$.

Let us be more specific. By a *deformation* of some object Y we mean another, larger object $\mathcal{Y}$ along with a morphism $\pi: \mathcal{Y} \rightarrow S$ to some parametrising pointed object $(S, 0 \in S)$, such that $\mathcal{Y}_0 := \pi^{-1}(0) \cong Y$. We call $\mathcal{Y}_0$ the *central fibre* of the family π . When $S = \text{Spec } \mathbb{C}[x]/(x^2)$ is the double point, we call π a *first-order deformation*. Similarly, we have higher-order deformations over $\text{Spec } \mathbb{C}[x]/(x^n)$ and formal deformations over $\text{Spec } \mathbb{C}[[x]]$ – but note that a formal deformation does not imply that an actual deformation exists, which is essentially asking for a formal power series to converge.

For example, in the category of schemes or of analytic spaces, a very popular deformation is a *flat smoothing*, which means that π is a flat morphism (which is a homological condition) and that the non-central fibres $\mathcal{X}_s$, $s \neq 0$ are smooth. If X is not smooth and a flat smoothing exists, then one can replace the study of the complicated object $\mathcal{X}_0$ by that of a smooth object $\mathcal{X}_s$, as long as one is concerned with properties that are invariant under flat deformations (like the Hilbert polynomial).

1.4 Some results from deformation theory

Remark 1.5. If $X \subset W$ is a subspace such that the conormal sheaf $N_{X,W}^*$ is ample, the deformation space of a bundle on $\hat{X}$ is finite-dimensional:

Fix an integer $m \geq 0$, a vector bundle E_m on $X^{(m)}$ and set $E_0 := E_m|_X$. If

$$h^2(X; \mathcal{E}nd E_0 \otimes S^m(N_{X,W}^*)) = 0,$$

then there exists a vector bundle E_{m+1} on $X^{(m+1)}$ such that $E_{m+1}|_{X^{(m)}} \cong E_m$ ([Pet81, Satz 1]).

Now let F be a vector bundle over X such that $h^2(X; \mathcal{E}nd F \otimes S^t(N_{X,W}^*)) = 0$ for all $t > 0$. If $N_{X,W}^*$ is ample, then $h^1(X; \mathcal{E}nd F \otimes S^t(N_{X,W}^*)) = 0$ for $t \gg 0$, and hence

$$\gamma = \sum_{t \geq 0} \gamma_t = \sum_{t \geq 0} h^1(X; \mathcal{E}nd F \otimes S^t(N_{X,W}^*)) < +\infty. \quad (1.1)$$

Then there exists a vector bundle G on $\hat{X}$ such that $G|_X \cong F$, and for a fixed such G the *deformation space* of G is isomorphic to $\mathbb{C}^\gamma$ ([Pet82, Satz 2], and first Bemerkung at p. 115, and see also [dJP00, Theorem 10.3.16]). There is a vector bundle A on an analytic neighbourhood U of X in W such that $A|_{\hat{X}} = G$, and hence $A|_X \cong F$ ([Pet82, Satz 3]).

Chapter 2

Surfaces

2.1 Introduction

In this chapter we focus on complex surfaces that contain an embedded line with negative self-intersection number. We have in mind the situation where the line $\ell \cong \mathbb{P}^1$ inside the surface Z has the conormal sheaf $\mathcal{N}_{\ell/Z}^* \cong \mathcal{O}_{\mathbb{P}^1}(-k)$. Since we are only interested in the local model, we will in fact *assume* that our space is the total space of a line bundle over $\mathbb{P}^1$ and define the spaces $Z_k := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$.

2.2 Digression: Application to mathematical physics

This section illustrates an application of our study of moduli to mathematical physics and the theory of instantons, but it is not necessary for the remainder of this thesis.

The complex dimension 2 is special in the sense that a complex 2-manifold is also a real 4-manifold, and the geometry of real 4-manifolds is famously very special. In the present case, we employ the *Kobayashi-Hitchin correspondence* for a compact Kähler manifold X of complex dimension 2. A unitary, anti-self-dual connection ∇ on a smooth $SU(2)$ -bundle $E \rightarrow X$ (i.e. an *instanton*) decomposes as $\nabla = \partial + \bar{\partial}$ with respect to the complex structure on X , such that $\bar{\partial}$ induces a holomorphic structure on E . The Kobayashi-Hitchin correspondence states that the map $\nabla \rightarrow \bar{\partial}$ is invertible and provides a one-to-one correspondence between $SU(2)$ -instantons and holomorphic bundles with vanishing first Chern class on X , and the instanton charge corresponds to $c_2(E)$. The correspondence has been proved in the cases when X is a projective surface by Donaldson and when X is compact Kähler by Uhlenbeck and Yau. In the non-compact case Donaldson proved the correspondence for $X = \mathbb{C}^2 = Z_0$ and King for the case where X is the blow-up of $\mathbb{C}^2$ at the origin, which we denote by Z_1 . In the non-compact cases, an instanton on X has to be understood as an instanton on the projective closure of X , which is $\mathbb{C}P^2$ in the case of $X = \mathbb{C}^2$ and the first Hirzebruch surface $\Sigma_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})$, with the additional condition that the bundle be trivial on a neighbourhood of the line at infinity. Of course in the non-compact case the second Chern class and the instanton charge vanish.

With this in mind it turns out that a holomorphic rank-2 bundle E on Z_k corresponds to an *instanton* if it extends to a bundle on the Hirzebruch surface $\Sigma_k := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1})$ such that the extension is trivial on the complement of ℓ , and we call E a *framed instanton* if a trivialisation of E on $Z_k \setminus \ell$ has been fixed. We will find certain numerical invariants for the description of the local moduli of bundles on Z_k , from which we can build a *local holomorphic Euler characteristic*, and we have proved that a version of the Kobayashi-Hitchin correspondence extends to the spaces Z_k and relates $SU(2)$ -instantons with *local charge* n

on Z_k to holomorphic rank-2 bundles with vanishing first Chern class and local holomorphic Euler characteristic n ([GKM08, Prop]). Here the local charge of an instanton E is thought of as the second Chern class of the trivial extension of E to Σ_k . It is “local” because the only contribution to this Chern class comes from a neighbourhood of ℓ , thanks to the following observation.

Under certain conditions that we make precise below, bundles on our model spaces Z_k can be “glued into” instantons on a larger (compact) surface X containing a line $Y \subset X$ such that $\mathcal{N}_{Y/X}^* \cong \mathcal{O}_{\mathbb{P}^1}(-k)$ by a process that was called “holomorphic surgery” in [GKM08]. This process works as follows. If E is a given instanton on X and we replace E by another bundle E' with $c_1(E') = 0$ such that $E|_{X \setminus Y} \cong E'|_{X \setminus Y}$, we have performed “holomorphic surgery”. If in addition $c_2(E') < c_2(E)$, we say that E has *decayed* to E' . The charge difference $c_2(E) - c_2(E')$ should be visible entirely locally near Y , and indeed it is.

If we write $N(Y)$ for a small, analytic neighbourhood of Y , then rank-2 bundles on $N(Y)$ can be identified with rank-2 bundles on Z_k (see Remark 2.3, it is *not* true that any tubular neighbourhood of Y is biholomorphic to Z_k), and on Z_k an instanton can indeed decay to the trivial bundle. Plugging this back into X , we say that E should be allowed to *decay locally*. If E' denotes the outcome of total local decay near Y , then $c_2(E) - c_2(E')$ is precisely the *local charge* of E near Y . It is the physical assumption that an instanton should locally be able to decay entirely that leads us to assume that E is trivial on $N(Y) \setminus Y$. In this case, holomorphic surgery works simply by fixing a framing of E on $N(Y) \setminus Y$ and identifying it with a framing on $N(\ell) \setminus \ell$ of the desired local instanton on Z_k .

To summarise, the study of the local situation on Z_k allows us to describe instanton decay on any compact surface that contains negative lines via the contribution of local instanton charges near those lines, which we model with the spaces Z_k .

2.3 Vector bundles on Z_k

The physics of the previous section is a In order to justify several of the constructions from the previous section, we must understand what holomorphic vector bundles on Z_k look like. Suppose then that $E \rightarrow Z_k$ is a holomorphic bundle. By the Grothendieck splitting principle, $E|_\ell \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$, and $c_1(E) = \sum_i a_i$. It turns out that in fact E is algebraically filtered, that is, made up from iterated algebraic extensions of bundles.

Theorem 2.1 ([Gas97]). *A holomorphic vector bundle $E \rightarrow Z_k$ of rank r is algebraically filtered, i.e. there exists an increasing filtration $E_1 \subset \dots \subset E_r = E$ such that E_1 is a line bundle and E_i/E_{i-1} is a line bundle for $2 \leq i \leq r$, and moreover all bundles E_i are algebraic.*

The spaces Z_k are special model spaces, and in fact this result works in much greater generality.

Theorem 2.2 (Ballico, Gasparim, Köppe). *Let W be a connected, complex manifold and $\ell \subset W$ a reduced, connected curve that is locally a complete intersection. If the conormal bundle $N_{\ell,W}^*$ is ample, then every vector bundle on $\widehat{\ell}$ is filtrable. If in addition ℓ is smooth, then every holomorphic bundle on $\widehat{\ell}$ is algebraic.*

Remark 2.3. In fact, every vector bundle on $\widehat{\ell}$ is determined already on a finite infinitesimal neighbourhood $\ell^{(m)}$. This is the reason that for the purpose of instanton decay we were allowed to identify bundles on $N(Y)$ and Z_k earlier.

Now we specialise to the case of rank-2 bundles. First note that $\text{Pic } Z_k \cong H^2(Z_k; \mathbb{Z}) \cong \mathbb{Z}$ and thus line bundles on Z_k are uniquely determined by their first Chern class, and they are

simply the pull-back of $\mathcal{O}_{\mathbb{P}^1}(r)$ from $\mathbb{P}^1$. Now if $E \rightarrow Z_k$ is a bundle of rank 2 with $c_1(E) = 0$, then by Grothendieck's splitting principle again, $E|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$, and we call the integer j the *splitting type* of E . Now by Theorem 2.1, E fits into a short exact sequence

$$0 \longrightarrow \mathcal{O}(-j) \longrightarrow E \longrightarrow \mathcal{O}(j) \longrightarrow 0. \quad (2.1)$$

We will also fix once and for all local coordinate charts on Z_k . Since Z_k is the total space of a vector bundle over the Riemann sphere $\mathbb{P}^1$, we only need two charts: Let $U \cong \mathbb{C}^2 = \{z, u\}$ and $V \cong \mathbb{C}^2 = \{z^{-1}, z^k u\}$. The bundle E is thus uniquely determined by one transition function on the overlap $U \cap V$, which can be expressed in the form

$$T = \begin{pmatrix} z^j & p(z, u) \\ 0 & z^{-j} \end{pmatrix}, \quad (2.2)$$

where p is a polynomial in z, z^{-1} and u .

2.4 Numerical invariants

The study of bundles on Z_k becomes more interesting when one considers the contraction $\pi: Z_k \rightarrow X_k$ of the zero section ℓ . To motivate this, let us return for a moment to the case of instantons on Z_1 and Σ_1 . Here the contraction maps Z_1 to $\mathbb{C}^2$, and Σ_1 to $\mathbb{P}^2$. Since the target is smooth, the direct image sheaf $\pi_* E$ of an instanton E is a sum of a locally free sheaf and torsion, so its double dual $(\pi_* E)^{\vee\vee}$ is locally free. On the compact spaces we can thus consider the difference $c_2(E) - c_2((\pi_* E)^{\vee\vee})$, which is nothing but the local charge of E that we met earlier. We compute this quantity directly by an application of Riemann-Roch and find

$$c_2(E) - c_2((\pi_* E)^{\vee\vee}) = h^0(X; (\pi_* E)^{\vee\vee} / \pi_* E) + h^0(X; R^1 \pi_* E). \quad (2.3)$$

The notion of a Chern class of a holomorphic bundle is well-defined on smooth manifolds, but on singular spaces there exist several inequivalent notions of Chern classes. However, the right-hand side of Equation 2.3 is independent of any notion of Chern class. In fact, it is a special case of what Blache [Bla96] defines as the *local holomorphic Euler characteristic* χ^{loc} of a reflexive sheaf near an isolated quotient singularity: Let $\sigma: (X, A) \rightarrow (X', x)$ be a resolution of an isolated quotient singularity and $\mathcal{F}$ a reflexive sheaf on X . Then

$$\chi^{\text{loc}}(\mathcal{F}, A, \sigma) := h^0(X'; (\sigma_* \mathcal{F})^{\vee\vee} / \sigma_* \mathcal{F}) + \sum_{i=1}^{n-1} (-1)^{i-1} h^0(X'; R^i \sigma_* \mathcal{F}). \quad (2.4)$$

For the case when X' is an orbifold, Blache [Bla96] shows that,

$$\chi(X, \mathcal{F}) = \chi(X', (\sigma_* \mathcal{F})^{\vee\vee}) + \sum_{x \in \text{Sing } X'} \chi^{\text{loc}}(\mathcal{F}, \sigma^{-1}(x), \sigma),$$

so the local holomorphic Euler characteristic measures precisely the amount of total Euler characteristic that is lost by contracting the orbifold resolution, or in other words the contribution from a neighbourhood of the exceptional set A .

Our spaces Z_k have cohomological dimension 1, so all higher derived images $R^i \pi_* E$ vanish for $i > 1$. For the smooth case Z_1 we have thus

$$c_2^{\text{loc}}(\ell, E) = \chi(\ell, E),$$

and from here we *define* the *local charge of E near ℓ* to be $\chi(\ell, E)$. We name the two con-

stituent summands the *width* $w_k(E)$ and the *height* $h_k(E)$ of the bundle E ,

Definition 2.4.

$$\chi^{\text{loc}}(\ell, E, \pi) = h^0(X_k; (\pi_* E)^{\vee\vee} / \pi_* E) + h^0(X_k; R^1 \pi_* E) = w_k(E) + h_k(E),$$

i.e.

$$w_k(E) := h^0(X_k; (\pi_* E)^{\vee\vee} / \pi_* E) \text{ and} \quad (2.5)$$

$$h_k(E) := h^0(X_k; R^1 \pi_* E). \quad (2.6)$$

Remark 2.5. The width $w_k(E)$ measures how far the direct image sheaf $\pi_* E$ is from being a split extension; the height $h_k(E)$ measures how close E is to being the split bundle (which is the unique bundle with maximal $w_k(E) + h_k(E)$ for a fixed j).

2.5 Bounds on the numerical invariants

The following results were proved in [BGK].

Theorem 2.6. *Let E be a rank-2 bundle over Z_k of splitting type j . Then the following bounds are sharp: For $j > 0$ and with $n_2 = \lfloor \frac{j}{k} \rfloor$,*

$$0 \leq w_k(E) \leq (j+1)n_2 - kn_2(n_2+1)/2, \text{ and } w_1(E) \geq 1.$$

Furthermore, for all $0 < j < k$, $w_k(E) = 0$ for all bundles E (and necessarily $k > 1$).

Proposition 2.7. *Let $E(j, p)$ be the bundle of splitting type j whose extension class is given by p , and let $\bar{E}(j) := \mathcal{O}(-j) \oplus \mathcal{O}(j)$ denote the split bundle. If $u|p(z, u)$ and $p \neq 0$, then*

$$h_k(\bar{E}(j)) \geq h_k(E(j, p)).$$

Corollary 2.8. *Let E be a rank-2 bundle over Z_k of splitting type j with $j > 0$ and let $j = nk + b$ as above. The following are sharp bounds for the local holomorphic Euler characteristic of E :*

$$j-1 \leq \chi(\ell, E) \leq \begin{cases} n^2 k + 2nb + b - 1 & \text{if } k \geq 2 \text{ and } 1 \leq b < k, \\ n^2 k & \text{if } k \geq 2 \text{ and } b = 0, \end{cases}$$

and

$$j \leq \chi(\ell, E) \leq j^2 \text{ for } k = 1.$$

2.6 Moduli

We would like to know the structure on the space of rank-2 bundles on Z_k . We already know from Theorem 2.1 that all such bundles are extensions of the form (2.1). For each fixed splitting type j , the space of such extensions is¹ $\text{Ext}_{Z_k}^1(\mathcal{O}(j), \mathcal{O}(-j)) \cong H^1(Z_k; \mathcal{O}(-2j))$.

Remark 2.9. Note that the space $\text{Ext}_{Z_k}^1(\mathcal{O}(j), \mathcal{O}(-j))$ is finite-dimensional for every j . Hence holomorphic bundles on Z_k are *algebraic*. Yet another way to see this is to note that a bundle

¹The isomorphism is, for any locally free $\mathcal{O}$ -module $\mathcal{L}$ of finite rank (see [Har77, Props. 6.7 and 6.3]),

$$\text{Ext}_{\mathcal{O}}^i(\mathcal{L}, -) := R^i \text{Hom}_{\mathcal{O}}(\mathcal{L}, -) \cong R^i \text{Hom}_{\mathcal{O}}(\mathcal{O}, - \otimes \mathcal{L}^{\vee}) \cong R^i \Gamma(- \otimes \mathcal{L}^{\vee}) =: H^i(- \otimes \mathcal{L}^{\vee}).$$

E is already determined on a *finite* infinitesimal neighbourhood $\ell^{(m)}$, which is a projective scheme and thus automatically satisfies GAGA.

After choosing coordinates, we can compute the space of extensions explicitly and find that it can be described as the space of coefficients of the polynomial p that appears in the transition function T in (2.2) for the bundle E . Precisely, this polynomial may be chosen to be of the *canonical form*

$$p(z, u) = \sum_{r=1}^{\lfloor (2j-2)/k \rfloor} \sum_{s=kr-j+1}^{j-1} p_{rs} z^s u^r. \quad (2.7)$$

Note that in this form, p is always divisible by u , which means that the restriction of E to ℓ splits as $\mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$. We may also consider the slightly more general form

$$\tilde{p}(z, u) = \sum_{r=0}^{\lfloor (2j-2)/k \rfloor} \sum_{s=kr-j+1}^{j-1} p_{rs} z^s u^r,$$

but note that when p is not divisible by u , then the splitting type of E is in fact lower than j . This slightly more general notion will be useful from the point of view of deformation theory below. See [Gas97, Theorem 3.3] for a proof of the general form.

The space of extensions does not solve the moduli problem for vector bundles, as different extensions may define holomorphically equivalent bundles. Isomorphic bundles must have the same splitting type, so we start by considering the spaces $\text{Ext}^1(\mathcal{O}(j), \mathcal{O}(-j))/\sim$, where $\sim$ denotes bundle isomorphism. These spaces will have a very complicated topology, but they give us an explicit handle. Part of this thesis consists of providing a means to decompose these spaces into finer components that possess a manifold structure.

From the explicit description of p we see that a bundle E splits on the m^{th} infinitesimal neighbourhood $\ell^{(m)}$ if and only if p is a multiple of u^{m+1} . Generically a bundle will not split already on the first neighbourhood, and this is the most important case we shall consider. On the first neighbourhood, we have a result.

Proposition 2.10. *If p and p' are two polynomials determining respectively two bundles E and E' on Z_k of splitting type j , then $E|_{\ell(1)} \cong E'|_{\ell(1)}$ if and only if $p = \lambda p'$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.*

Making the notion of genericity precise amounts to finding a suitable notion of *stability* on the moduli of bundles.

Definition 2.11. For each integer j and each k we define

$$\mathfrak{M}(Z_k; j) := \text{Ext}^1(\mathcal{O}(j), \mathcal{O}(-j))/\sim$$

to be the space of extensions of $\mathcal{O}(j)$ by $\mathcal{O}(-j)$ on Z_k up to bundle isomorphism. If $E \in M_{Z_k}(j)$ is such an extension, we call E *generic* if the width $w_k(E)$ and the height $h_k(E)$ of E attain the minimal value.

Remark 2.12. From the previous section we know the precise value of the lower bound. This bound is always realised on all spaces Z_k by the bundle determined by $p(z, u) = zu$.

Theorem 2.13. *For $j \geq k$, $\mathfrak{M}(Z_k, j)$ has an open, dense subspace homeomorphic to a complex projective space $\mathbb{P}^{2j-2-k}$ minus a closed subvariety of codimension at least $k+1$.*

It is possible to embed the moduli $\mathfrak{M}(Z_k, j)$ into $\mathfrak{M}(Z_k, j+k)$ via two *elementary transformations* (which we explain presently) and a twist: Let $\Phi: \mathfrak{M}(Z_k; j) \rightarrow \mathfrak{M}(Z_k; j+k)$ be defined

by

$$\Phi(E) = \text{Elm}_{\mathcal{O}_\ell(j+k)}(\text{Elm}_{\mathcal{O}_\ell(j)}(E)) \otimes \mathcal{O}(-k).$$

In coordinates, Φ sends the bundle given by (j, p) to $(j + k, z^k u^2 p)$.

Theorem 2.14. *The map Φ is well defined, injective and a homeomorphism onto its image, which consists of all bundles in $\mathfrak{M}(Z_k; j + k)$ that split on the second infinitesimal neighbourhood of ℓ .*

Open question: How does Φ affect the numerical invariants?

2.6.1 Elementary transformations

An important tool in the study of vector bundles and sheaves is the elementary transformation, which changes a locally free sheaf over a divisor. It works as follows. Let W be an algebraic variety, $D \subset W$ a Cartier divisor and $\mathcal{L} \in \text{Pic}(D)$ a fixed line bundle on D . If $\mathcal{E}$ is any locally free sheaf on W and $r: \mathcal{E} \rightarrow \mathcal{L}$ a surjection that is induced by a surjection $\rho: \mathcal{E}|_D \rightarrow \mathcal{L}$, then $\mathcal{E}' := \mathcal{K}er(r)$ is called the *elementary transformation* of $\mathcal{E}$ induced by r , written $\text{Elm}_{\mathcal{L}}(\mathcal{E})$. Since the divisor D is Cartier, $\mathcal{E}'$ is locally free. Writing $\mathcal{L}' := \mathcal{K}er(\rho)$, we obtain the *display* of the elementary transformation:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{L}' & \longrightarrow & \mathcal{E}|_D & \xrightarrow{\rho} & \mathcal{L} \longrightarrow 0 \\ & & \uparrow t & & \uparrow & & \parallel \\ 0 & \longrightarrow & \mathcal{E}' & \xrightarrow{F} & \mathcal{E} & \xrightarrow{r} & \mathcal{L} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \\ & & \mathcal{E}(-D) & \xlongequal{\quad} & \mathcal{E}(-D) & & \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

Note that the induced surjection $t: \mathcal{E}' \rightarrow \mathcal{L}'$ gives the inverse elementary transformation (up to twisting by D).

In local coordinates. Our spaces Z_k have one compact divisor $\ell \cong \mathbb{P}^1$ given by

$$0 \longrightarrow \mathcal{O}_{Z_k}(-k) \longrightarrow \mathcal{O}_{Z_k} \longrightarrow \mathcal{O}_\ell \longrightarrow 0,$$

In our canonical local coordinates, ℓ is given by $\{u = 0\}$ on the chart U and by $\{z^k u = 0\}$ on the chart V , so the left map is just multiplication by u or $z^k u$ on the respective charts. Since every rank-2 bundle comes with a surjection $\mathcal{E} \rightarrow \mathcal{O}(j)$, the restriction to ℓ gives a surjection

$$r: \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^1}(j),$$

and we are in a position to apply an elementary transformation with respect to r to the bundle $\mathcal{E}$. In coordinates, r maps a local section (a, b) to the residue of b modulo (u) on the U -chart. The kernel of r (which is $\mathcal{E}'$) thus consists of all sections (a, b) for which b vanishes

on ℓ . If $\mathcal{E}$ is given by (2.1), then $\mathcal{E}'$ is an extension is an extension

$$0 \longrightarrow \mathcal{O}(-j) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{O}(j+k) \longrightarrow 0,$$

Thus $\mathcal{E}'$ has transition function

$$T' = \begin{pmatrix} z^j & p' \\ 0 & z^{-j-k} \end{pmatrix},$$

and the inclusion $F = (f, \tilde{f}): \mathcal{E}' \rightarrow \mathcal{E}$ is given by $f(a, b) = (a, ub)$ on U and $\tilde{f}(A, B) = (A, z^k uB)$ on V . Since we must have $T \circ f = \tilde{f} \circ T'$, we compute

$$T \circ f \begin{pmatrix} a \\ b \end{pmatrix} = T \begin{pmatrix} a \\ ub \end{pmatrix} = \begin{pmatrix} z^j a + upb \\ z^{-j} ub \end{pmatrix} \quad \text{and} \quad \tilde{f} \circ T' \begin{pmatrix} a \\ b \end{pmatrix} = \tilde{f} \begin{pmatrix} z^j a + p'b \\ z^{-j-k} ub \end{pmatrix} = \begin{pmatrix} z^j a + p'b \\ z^{-j} ub \end{pmatrix},$$

and thus $p' = up$.

The new bundle $\mathcal{E}'$ now comes with a surjection to $\mathcal{O}_{\mathbb{P}^1}(j+k)$, so we can perform another elementary transformation to arrive at a bundle $\mathcal{E}''$ with transition function

$$T'' = \begin{pmatrix} z^j & u^2 p \\ 0 & z^{-j-2k} \end{pmatrix}.$$

Finally, $\mathcal{E}''(-k)$ is a rank-2 bundle with vanishing first Chern class and splitting type $j+k$, and we see that the map $\mathcal{E} \rightarrow \mathcal{E}''(-k)$ is given in coordinates by $p(z, u) \mapsto z^k u p(z, u)$.

For completeness, we record that the inverse transformation is given by the surjection $t: \mathcal{E}' \rightarrow \mathcal{L}' \cong \mathcal{O}_{\mathbb{P}^1}(-j)$. The map t is given on the U -chart by mapping (a, b) to the residue of a modulo (u) , and on the V -chart by mapping $(\tilde{a}, \tilde{b})$ to the residue of $\tilde{a}$ modulo $(z^k u)$.

2.7 Stability via the endomorphism bundle

Classical deformation theory of vector bundles on a (compact) surface X says that the obstruction to deforming a bundle $E \rightarrow X$ live in the second cohomology $H^2(X; \mathcal{E}nd E)$ (see [FK74], and the moduli space is smooth if this obstruction vanishes. In this case, the tangent space to the moduli space at E is the space of first-order deformations of E , $H^1(X; \mathcal{E}nd E)$, modulo the space of trivial deformations (i.e. deformations into isomorphic bundles) $H^0(X; \mathcal{E}nd E)$.

In our case we take $X = Z_k$, which is of cohomological dimension one, so the second cohomology of all coherent sheaves vanishes. Since Z_k is not compact, we cannot conclude that the moduli of vector bundles is a smooth space, and we already saw that this is not the case even for $SL(2)$ -bundles. However, the bundle (or sheaf) $\mathcal{E}nd E$ still contains valuable numerical information, which in fact turns out to be equivalent to the information given by the width and height for instanton bundles. However, this perspective offers another interpretation of the invariants, and we may ask for a physical interpretation of the non-instanton bundles.

To be precise, we define two numbers that we will suggestively call h^1 and h^0 . This notation is concise at the risk of being confusing, but the context should make clear what is meant. First off, since Z_k is the total space of a negative bundle over $\mathbb{P}^1$, the cohomology of $\mathcal{E}nd E$ vanishes in degrees ≥ 2 and is finite-dimensional in degree 1. Next we consider the zeroth cohomology of $\mathcal{E}nd E$. It is infinite-dimensional, since Z_k is non-compact and H^0 is the space of global sections. However, the *difference* of dimensions of H^0 for two different bundles is finite in a certain sense: Consider the restriction of E to the m^{th} infinitesimal neighbourhood of ℓ . This space is projective and so $V_m(E) := H^0(\ell^{(m)}; \mathcal{E}nd(E)|_{\ell^{(m)}})$ is finite-dimensional, although the dimension of this space grows with m . But for each fixed m ,

we can compare the dimensions of $V_m(E)$ and $V_m(E_{\text{split}})$, where we write E_{split} for the split bundle of the same splitting type as E , so $E_{\text{split}} \cong \mathcal{O}(j) \oplus \mathcal{O}(-j)$. This difference of dimensions is independent of m for large values m . Thus we define

$$\begin{aligned} h^1(E) &:= h^1(Z_k; \mathcal{E}nd(E)), \text{ and} \\ h^0(E) &:= h^0(\ell^{(m)}; \mathcal{E}nd(E_{\text{split}})|_{\ell^{(m)}}) - h^0(\ell^{(m)}; \mathcal{E}nd(E)|_{\ell^{(m)}}), \end{aligned}$$

where m is taken sufficiently large so that the expression for $h^0(E)$ stabilises, which happens for $m \geq (4j - 2)/k$.

While the numbers $h^0(E)$ and $h^1(E)$ are analytic invariants of E , they are in fact equivalent to $w(E)$, $h(E)$ on instanton bundles, where $j = nk$ for some n , via the following relations:

$$\begin{aligned} w(E) + h(E) &= \chi(E) = ((h^1(E) - h^0(E)) - j)/2 + j/k \\ h^0(E) + h^1(E) &= h^1(\mathcal{E}nd(E_{\text{split}})) \end{aligned}$$

This gives

$$h^0(E) = n^2 k - \chi, \quad h^1(E) = kn(n+1) - 2n + \chi(E).$$

For non-instanton bundles, though, the numbers $h^0(E)$ and $h^1(E)$ (at least one of them, as they are not independent) provide additional information on top of $w(E)$, $h(E)$.

2.A Sample computation

We compute explicitly the width and height for a simple, non-trivial example, namely of the bundle E of splitting type $j = 3$ on the space $Z_2 = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2))$ given by $p(z, u) = u$, so E has transition matrix

$$T = \begin{pmatrix} z^3 & u \\ 0 & z^{-3} \end{pmatrix}.$$

The space Z_2 has coordinate charts $U = \{(z, u)\}$ and $V = \{(z^{-1}, z^2 u)\}$. The contraction of ℓ is

$$\pi: Z_2 \rightarrow X_2 = \text{Spec } R, \text{ where } R = \mathbb{C}[x_0, x_1, x_2]/(x_0 x_2 - x_1^2).$$

Width. To compute $Q = (\pi_* E)^{\vee\vee}/\pi_* E$, we first compute sections of E over $\ell^{(n)}$ for all n . This amounts to computing the space of sections (a, b) of E as *formal* power series $a, b \in \mathbb{C}[[z, u]]$, subject to the condition that a, b be holomorphic in $\{z, u\}$ and $z^{-3}b, z^3a + ub$ be holomorphic in $\{z^{-1}, z^2 u\}$. This implies that b has the following form:

$$\begin{aligned} b(z, u) &= b_{00} + b_{01}z + b_{02}z^2 + b_{03}z^3 + \cancel{b_{04}z^4} + \dots \\ &+ b_{10}u + \dots + b_{15}z^5u + \cancel{b_{16}z^6u} + \dots \\ &+ b_{20}u^2 + \dots + b_{27}z^7u^2 + \cancel{b_{28}z^8u^2} + \dots \\ &+ \dots \end{aligned}$$

All terms $z^s u^r$ in b with $s - 3 > 2r$ have to vanish. Now we can compute an expression for a . Since z^{s+3} is never holomorphic in $\{z^{-1}, z^2 u\}$ for $s \geq 0$, there are no terms on $\ell^{(0)}$. We are left

with the following:

$$\begin{aligned} a(z, u) &= \cancel{a_{00}} + \cancel{a_{01}u} \\ &+ a_{10}u + a_{11}zu + \cdots \\ &+ a_{20}u^2 + a_{21}zu^2 + a_{22}z^2u^2 + \cdots \\ &+ \cdots \end{aligned}$$

Now we consider the expression $z^3a + ub$ and pick out every coefficient that must vanish, i.e. where $\deg_z > 2 \deg_u$. The first few such terms are:

$$\begin{aligned} z^3u(a_{10} + b_{03}) &= 0 & z^4u a_{11} &= 0 & z^5u a_{12} &= 0 & \cdots \\ z^5u^2(a_{22} + b_{15}) &= 0 & z^6u^2 a_{23} &= 0 & \cdots \end{aligned}$$

Finally we can write down generators:

$$\beta_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \beta_1 = \begin{pmatrix} 0 \\ z \end{pmatrix} \quad \beta_2 = \begin{pmatrix} 0 \\ z^2 \end{pmatrix} \quad \alpha = \begin{pmatrix} u \\ -z^3 \end{pmatrix} \quad \alpha' = \begin{pmatrix} u^2 \\ 0 \end{pmatrix}$$

We could have written down many more generators, but *over the space* X_2 , i.e. over the ring

$$R = \{x_0 = u, x_1 = zu, x_2 = z^2u\} / (x_0x_2 - x_1^2),$$

everything else can be expressed in terms of $\beta_0, \beta_1, \beta_2, \alpha$. (For example, $\alpha' = x_0\alpha + x_1\beta_2$.) It remains to find the relations among the generators, and we arrive at the complete description of the R -module

$$M = \langle \beta_0, \beta_1, \beta_2, \alpha \rangle_R / (x_1\beta_0 - x_0\beta_1, x_2\beta_0 - x_1\beta_1, x_1\beta_1 - x_0\beta_2, x_2\beta_1 - x_1\beta_2).$$

Application of the Theorem on Formal Functions tells us (in a highly non-trivial fashion) that $Q \cong \text{coker } M \xrightarrow{\text{ev}} M^{\vee\vee}$. Thus we must compute $M^\vee$ and thence $M^{\vee\vee}$. A moment's thought shows:

$$M^\vee = \text{Hom}_R(M, R) = \{\beta^\vee, \alpha^\vee\},$$

where

$$\beta^\vee = \{\beta_i \mapsto x_i, \alpha \mapsto 0\} \quad \text{and} \quad \alpha^\vee = \{\beta_i \mapsto 0, \alpha \mapsto 1\}.$$

$M^\vee$ is already free, so $M^{\vee\vee}$ free as well, given by

$$M^{\vee\vee} = \{\beta^{\vee\vee} = \{\beta^\vee \mapsto 1, \alpha^\vee \mapsto 0\}, \alpha^{\vee\vee} = \{\beta^\vee \mapsto 0, \alpha^\vee \mapsto 1\}\}.$$

The evaluation map $\text{ev}: M \rightarrow M^{\vee\vee}$ acts as follows:

$$\text{ev}(\alpha) = \alpha^{\vee\vee} \quad \text{ev}(\beta_i) = x_i \beta^\vee$$

Thus the only element in $M^{\vee\vee}$, seen as a $\mathbb{C}$ -vector space, that is *not* in the image of ev is the element $1 \cdot \beta^{\vee\vee}$, so $\text{coker}(\text{ev}) = \langle \beta^{\vee\vee} \rangle_{\mathbb{C}}$, which has dimension one, so $w(E) = 1$.

Height. The height of E is $h(E) := h^0(X_2; R^1\pi_*E)$. But $h^0 = \dim H^0$ is just the dimension of the stalk $(R^1\pi_*E)_0$. Now the dimension is the same for the stalk of the sheaf and the stalk of the completion of the sheaf, and the latter is computed by the Theorem on Formal Functions:

$$\dim(R^1\pi_*E)_0 = \dim(R^1\pi_*E)_0^\wedge = \dim\left(\varprojlim_n H^1(\ell^{(n)}; E|_{\ell^{(n)}})\right). \quad (2.8)$$

Since the limit stabilises at finite n , the computation is actually easy and amounts to computing $H^1(\widehat{\mathcal{L}}; E)$, which we will do now.

An element of H^1 is a section over $U \cap V$ modulo holomorphic sections over U or over V . We simply write down all such sections. In our case we have exactly two of them:

$$\begin{pmatrix} z^{-1} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z^{-2} \\ 0 \end{pmatrix}$$

Thus $h(E) = 2$.

2.B Algorithmic computation of the invariants

[I can supply here a description of the general computational algorithms for the computation of height, width, h^0 and h^1 . This appears on my website <http://www.maths.ed.ac.uk/~s0571100/Instanton/> and on the *Macaulay 2* website <http://www.math.uiuc.edu/Macaulay2/>, but I can omit it if it is not of interest.]

Chapter 3

Threefolds

3.1 Introduction

In this chapter we study the local moduli problem on complex threefolds, and in the same way as in Chapter 2 we assume that our threefold W contains an embedded line $\mathbb{P}^1 \cong \ell \subset W$, and the normal bundle of ℓ will play a crucial role. Unlike in the previous chapter, ℓ is not of middle dimension and there is no analogue of the self-intersection number, but instead we will consider whether ℓ moves in W or whether it is rigid, or even infinitesimally rigid. After establishing these properties, we continue to study the local moduli of bundles on W near ℓ as before.

An important ingredient in the study of the local moduli is extent to which a version of the GAGA principle holds on the spaces in question. Since the spaces are local $\mathbb{P}^1$ s and GAGA holds on $\ell \cong \mathbb{P}^1$, we proceed by studying the infinitesimal neighbourhoods $\ell^{(m)}$ and the formal completion $\hat{\ell}$. We consider three different examples: On the first space, bundles are filtered and algebraic, on the second they are filtered but not necessarily algebraic, and on the third there are rank-2 bundles that are not extensions.

In § 3.2 we define the spaces of interest and derive some explicit descriptions. We proceed to discuss the GAGA property of these spaces in § 3.3 before turning to the moduli problem proper. Endomorphism bundles are discussed in § 3.4 and numerical invariants in § 3.5 in preparation for the description of the moduli of bundles in the final § 3.6.

3.2 Local Calabi-Yau threefolds with rational curves

As we are studying spaces with an embedded compact line $\mathbb{P}^1$, we reduce to the simplest such case, which is that of the total space of normal bundle of the line (also called a *local* $\mathbb{P}^1$ by algebraic geometers). The compact 1-cycles of such a space correspond to the holomorphic sections of the normal bundle, and they are all rationally equivalent to the zero section.

Local $\mathbb{P}^1$ s are particularly amenable as they can be covered by two charts only; thus we perform explicit calculations. Moreover, all coherent cohomology groups vanish for all degrees but 0 and 1.

Motivated by the question how the moduli of bundles changes under birational transformation of the base, we are keeping in mind the question of whether ℓ may be contracted.

3.2.1 Definitions

We restrict our attention to local $\mathbb{P}^1$ spaces $W \cong N_{\ell/W}$ that are Calabi-Yau. Since

$$c_1(W) = c_1(\mathbb{P}^1) + c_1(N_{\ell/W}) \quad \text{and} \quad N_{\ell/W} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b),$$

we have $a + b = -2$. When considering the contraction of a line inside a threefold, then according to [Jim92] only three essential local models may occur:

$$\begin{aligned} W_1 &:= \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \\ W_2 &:= \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}) \\ W_3 &:= \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \end{aligned}$$

In each case we denote by Z the zero-section, so that $Z \cong \mathbb{P}^1$. The spaces have canonical charts $U \cong \mathbb{C}^2 \cong \{z, u, v\}$ and V , where respectively $V \cong \mathbb{C}^2 \cong \{z^{-1}, zu, zv\}$, $\{z^{-1}, z^2u, v\}$ and $\{z^{-1}, z^3u, z^{-1}v\}$. In each case, the canonical bundle is spanned globally by $dz \wedge du \wedge dv$, so we see explicitly that the spaces are Calabi-Yau. Note that the conormal sheaves of W_2 and W_3 are not ample. (More generally, all the spaces $W_i := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-i) \oplus \mathcal{O}_{\mathbb{P}^1}(i-2))$ are Calabi-Yau, but we will not consider them.)

3.2.2 Canonical forms

Let now E be a bundle on one of the complex spaces W_i , $i = 1, 2, 3$ of splitting type j . Assume for now that E is an extension of line bundles

$$0 \longrightarrow \mathcal{O}(-j) \longrightarrow E \longrightarrow \mathcal{O}(j) \longrightarrow 0,$$

where $\mathcal{O}(j)$ is just the pullback of $\mathcal{O}_{\mathbb{P}^1}(j)$, given by the transition matrix

$$T = \begin{pmatrix} z^j & p(z, u, v) \\ 0 & z^{-j} \end{pmatrix}.$$

(It follows from Theorems 3.10 and 3.11 that the transition matrix of every bundle on W_1 and W_2 may be put in this form.) It is necessary that p be of the form $p(z, u, v) = up'(z, u, v) + vp''(z, u, v)$, for otherwise the bundle E would in fact be of lower splitting type. This is an important point to which we return in the discussion of deformation spaces.

Proposition 3.1. *The extension class p can be reduced to the following form, respectively,*

$$\begin{aligned} \text{on } W_1: \quad p(z, u, v) &= \sum_{t=\epsilon}^{2j-2} \sum_{r=1-\epsilon}^{2j-2-t} \sum_{s=r+t-j+1}^{j-1} p_{trs} z^s u^r v^t, \\ \text{on } W_2: \quad p(z, u, v) &= \sum_{t=\epsilon}^{\infty} \sum_{r=1-\epsilon}^{j-1} \sum_{s=2r-j+1}^{j-1} p_{trs} z^s u^r v^t, \text{ and} \\ \text{on } W_3: \quad p(z, u, v) &= \sum_{t=\epsilon}^{\infty} \sum_{r=1-\epsilon}^{\lfloor \frac{2j-2+t}{3} \rfloor} \sum_{s=3r-t-j+1}^{j-1} p_{trs} z^s u^r v^t, \end{aligned}$$

where $\epsilon \in \{0, 1\}$.

Definition 3.2 (Canonical extension class). We call the form of p from Proposition 3.1 the *canonical form* of the extension.

Proof of Proposition 3.1. Suppose that E is an extension given by the transition matrix

$$T = \begin{pmatrix} z^j & p(z, u, v) \\ 0 & z^{-j} \end{pmatrix}.$$

A priori, p is given by a convergent power series

$$p(z, u, v) = \sum_{t=\epsilon}^{\infty} \sum_{r=1-\epsilon}^{\infty} \sum_{s=-\infty}^{\infty} p_{trs} z^s u^r v^t,$$

where $\epsilon \in \{0, 1\}$ accounts for the vanishing of p on the zero section $u = v = 0$. A bundle isomorphism casts T into the new form

$$T' = \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z^j & p(z, u, v) \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (3.1)$$

where $\alpha, \beta, \gamma, \delta$ are holomorphic on V and A, B, C, D are holomorphic on U . In particular, we consider only $C = \gamma = 0$, whence $\alpha A = \delta D = 1$, and we can write

$$T' = \begin{pmatrix} \alpha & \beta \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} z^j & p(z, u, v) \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} \alpha^{-1} & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix},$$

with

$$p' = \alpha B z^j + \beta D z^{-j} + \alpha D p. \quad (3.2)$$

We may fix $\alpha = D = 1$, say, and use β and B to remove terms from the power series of p :

First, any term $p_{trs} z^s u^r v^t$ with $s \geq j$ can be removed from p by setting $\beta = 0$ and $B = -p_{trs} z^{s-j} u^r v^t$; B is holomorphic on U . Thus we only need terms with $s \leq j-1$. Secondly, for fixed r and t , by setting $B = 0$ and $\beta = -p_{trs} z^{s+j} u^r v^t$ we remove terms with

$$\begin{aligned} r + t - j &\leq s && \text{on } W_1, \\ 2r - j &\leq s && \text{on } W_2, \text{ and} \\ 3r - t - j &\leq s && \text{on } W_3. \end{aligned}$$

Finally, we have obtained constraints for s, r and t for the remaining terms in p as follows:

$$\begin{aligned} r + t - j + 1 \leq s \leq j - 1 &\Rightarrow r + t \leq 2j - 2 && \text{on } W_1, \\ 2r - j + 1 \leq s \leq j - 1 &\Rightarrow r \leq j - 1 && \text{on } W_2, \text{ and} \\ 3r - t - j + 1 \leq s \leq j - 1 &\Rightarrow 3r - t \leq 2j - 2 && \text{on } W_3. \end{aligned}$$

The result follows immediately. $\square$

Remark 3.3. If instead we also allow terms which are not multiples of u or v , we include extensions of lower splitting type. These more general functions are obtained by starting both sums over r and t at zero. We write, for example on W_1 ,

$$\tilde{p}(z, u, v) = \sum_{t=0}^{2j-2} \sum_{r=0}^{2j-2-t} \sum_{s=r+t-j+1}^{j-1} p_{trs} z^s u^r v^t,$$

which determines an extension of splitting type $\leq j$ (but is not in canonical form if the splitting type is strictly less than j).

Corollary 3.4. *If $\lambda \neq 0$, then p and λp determine isomorphic bundles.*

Proof. In Equation (3.2), let $\beta = B = 0$, $\alpha = 1$ and $D = \lambda$. □

On the first infinitesimal neighbourhood $Z^{(1)}$, the converse of Corollary 3.4 is true. We are working on the formal completion $\widehat{Z}$, so local section of the structure sheaf $\mathcal{O}_{\widehat{Z}}$ is a *formal* power series in u and v and *convergent* in z . Peternell's Existence Theorem (see Remark 1.5) asserts that a bundle on $\widehat{Z}$ extends to a holomorphic bundle on an actual open (in the analytic topology) neighbourhood of Z . Thus we are allowed to let the entries of the matrices in the isomorphism (3.1) be formal power series. It is always possible to choose a nowhere vanishing formal power series with a finite number of prescribed coefficients, so that we can always make sure that the coordinate change matrices have nowhere vanishing determinant.

Proposition 3.5 (Bundles on $Z^{(1)}$). *On the first infinitesimal neighbourhood $Z^{(1)}$ in any of the three spaces W_1 , W_2 or W_3 , the only isomorphism of bundles is scaling. That is, two bundles $E_1|_{Z^{(1)}}$ and $E_2|_{Z^{(1)}}$ given by transition matrices*

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z^j & q \\ 0 & z^{-j} \end{pmatrix}$$

are isomorphic if and only if $q = \lambda p$ for some $\lambda \in \mathbb{C}^\times$.

Proof. The “if”-part is just Corollary 3.4.

For the “only if”-part, first note that the restriction to the first neighbourhood $Z^{(1)}$ implies that p and q only contain powers of u and v of total degree 1. It follows from Proposition 3.1 that p and q only contain certain powers of z^s , namely those with

$$\left. \begin{array}{ll} 2-j \leq s \leq j-1 & \text{on } W_1, \\ \text{in } z^s u: 3-j \leq s \leq j-1 \\ \text{in } z^s v: 1-j \leq s \leq j-1 \end{array} \right\} \quad \text{on } W_2, \text{ and} \quad (3.3)$$

$$\left. \begin{array}{ll} \text{in } z^s u: 4-j \leq s \leq j-1 \\ \text{in } z^s v: -j \leq s \leq j-1 \end{array} \right\} \quad \text{on } W_3.$$

Next, the two bundles are isomorphic only if there exist matrices holomorphic on the respective charts such that

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z^j & q \\ 0 & z^{-j} \end{pmatrix}.$$

Here A, B, C, D are power series on $U \cap Z^{(1)}$ and $\alpha, \beta, \gamma, \delta$ on $V \cap Z^{(1)}$, and we write, for instance, $A(z, u, v) = a_{00}(z) + a_{10}(z)u + a_{01}(z)v$ etc., where the coefficients are power series in z , and similarly for respectively $\alpha(z^{-1}, zu, zv)$, $\alpha(z^{-1}, z^2u, v)$ and $\alpha(z^{-1}, z^3u, z^{-1}v)$.

Comparing the two sides of the equation term by term gives four equations. We will only go through the case of W_1 here; for the other two just replace zu and zv in the following by z^2u and v for W_2 or by z^3u and $z^{-1}v$ for W_3 .

$$(a_{00}(z) + a_{10}(z)u + a_{01}(z)v)z^j + pc_{00}(z) = (\alpha_{00}(z^{-1}) + \alpha_{10}(z^{-1})zu + \alpha_{01}(z^{-1})zv)z^j \quad (3.4)$$

$$z^{-j}(c_{00}(z) + c_{10}(z)u + c_{01}(z)v) = (\gamma_{00}(z^{-1}) + \gamma_{10}(z^{-1})zu + \gamma_{01}(z^{-1})zv)z^j \quad (3.5)$$

$$(b_{00}(z) + b_{10}(z)u + b_{01}(z)v)z^j + pd_{00}(z) = \alpha_{00}(z^{-1})q + (\beta_{00}(z^{-1}) + \beta_{10}(z^{-1})zu + \beta_{01}(z^{-1})zv)z^{-j} \quad (3.6)$$

$$z^{-j}(d_{00}(z) + d_{10}(z)u + d_{01}(z)v) = \gamma_{00}(z^{-1})q + (\delta_{00}(z^{-1}) + \delta_{10}(z^{-1})zu + \delta_{01}(z^{-1})zv)z^{-j} \quad (3.7)$$

The polynomials p and q are divisible by either u or v , so comparing the terms in (3.4) and (3.7) that are independent of both u and v gives $a_{00}(z) = \alpha_{00}(z^{-1})$ and $d_{00}(z) = \delta_{00}(z^{-1})$, whence all four are constants and $a_{00} = \alpha_{00}$ and $d_{00} = \delta_{00}$.

Next, equating terms in u or v in (3.6) gives

$$(b_{10}(z)u + b_{01}(z)v)z^j + pd_{00} = \alpha_{00}q + (\beta_{10}(z^{-1})zu + \beta_{01}(z^{-1})zv)z^{-j}.$$

By the conditions (3.3) on p and q , we must have

$$b_{10}(z)u + b_{01}(z)v = 0 \quad \text{and} \quad \beta_{10}(z^{-1})zu + \beta_{01}(z^{-1})zv = 0$$

(whence $b_{10} = 0 = b_{01}$ and $\beta_{10} = 0 = \beta_{01}$), and so $pd_{00} = \alpha_{00}q$.

The proof is finished by showing that $\alpha_{00}d_{00} \neq 0$. But the terms independent of both u and v in (3.6) yield $b_{00}(z)z^j = \beta_{00}(z^{-1})z^{-j}$, whence $b_{00} = 0 = \beta_{00}$. Thus over Z the coordinate change has determinant $a_{00}d_{00} = \alpha_{00}d_{00} \neq 0$. $\square$

Remark 3.6. Inspection of the proof of Proposition 3.1 shows that the conditions (3.3) and the equations (3.4)–(3.7) match up precisely, and that the conclusions of Proposition 3.5 are in fact valid for all spaces $W_i = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-i) \oplus \mathcal{O}_{\mathbb{P}^1}(i-2))$.

Remark 3.7. Another way of seeing extensions of the form (2.1) is by considering the isomorphism¹

$$\text{Ext}_{\mathcal{O}_{W_i}}^1(\mathcal{O}(j), \mathcal{O}(-j)) \cong H^1(W_i; \mathcal{O}(-j) \otimes \mathcal{O}(j)^\vee) \cong H^1(W_i; \mathcal{O}(-2j)). \quad (3.8)$$

Direct computation shows that this is precisely the space of all coefficients in the generalised extension form from Remark 3.3, and the space of extensions E that satisfy $E|_Z \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$ is thus precisely the subset of $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(j), \mathcal{O}_{\mathbb{P}^1}(-j))$ consisting of extension classes of the form $p(z, u, v) = up'(z, u, v) + vp''(z, u, v)$. Proposition 3.1 says that all terms in p that lie outside the given range are coboundaries with respect to this H^1 .

In fact, computations of H^1 -groups will be useful once more: The height of a rank-2 bundle E near an exceptional set, as defined by Equation (2.6), can be computed by the Theorem on Formal Functions (2.8) as follows:

$$\begin{aligned} h(E) &:= h^0(W'; R^1\pi_*E) = \dim H^0(W'; R^1\pi_*E) \\ &= \dim(R^1\pi_*E)_0 = \dim \left(\varprojlim_n H^1(Z^{(n)}; E) \right). \end{aligned}$$

But since E is algebraic in the cases which we consider (namely on W_1 and on the hypersurfaces D_i), the limit in the right-most term stabilises at finite n , and it remains to compute H^1 formally on $\hat{Z}$. To this end, we present a canonical form of 1-cocycles representing elements of $H^1(\hat{Z}; E)$:

¹The isomorphism is, for any locally free $\mathcal{O}$ -module $\mathcal{L}$ of finite rank (see [Har77, Props. 6.7 and 6.3]),

$$\text{Ext}_{\mathcal{O}}^i(\mathcal{L}, -) := R^i \text{Hom}_{\mathcal{O}}(\mathcal{L}, -) \cong R^i \text{Hom}_{\mathcal{O}}(\mathcal{O}, - \otimes \mathcal{L}^\vee) \cong R^i \Gamma(- \otimes \mathcal{L}^\vee) =: H^i(- \otimes \mathcal{L}^\vee).$$

Proposition 3.8 (Canonical cocycle). *Let E be an extension as in Proposition 3.1. A 1-cocycle $\sigma \in H^1(E)$ has the canonical representative, respectively,*

$$\begin{aligned} \text{on } W_1: \quad \sigma &= \sum_{t=0}^{j-2} \sum_{r=0}^{j-2-t} \sum_{s=r+t-j+1}^{-1} z^s u^r v^t \begin{pmatrix} a_{trs} \\ 0 \end{pmatrix}, \\ \text{on } W_2: \quad \sigma &= \sum_{t=0}^{\infty} \sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} \sum_{s=2r-j+1}^{-1} z^s u^r v^t \begin{pmatrix} a_{trs} \\ 0 \end{pmatrix}, \text{ and} \\ \text{on } W_3: \quad \sigma &= \sum_{t=0}^{\infty} \sum_{r=0}^{\lfloor \frac{t+j-2}{3} \rfloor} \sum_{s=3r-t-j+1}^{-1} z^s u^r v^t \begin{pmatrix} a_{trs} \\ 0 \end{pmatrix}. \end{aligned}$$

Remark 3.9. Cocycles in $H^1(D_i; E)$ are obtained from this by setting $v = 0$.

Proof of 3.8. A priori, σ is given by

$$\sigma = \begin{pmatrix} a \\ b \end{pmatrix} = \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=-\infty}^{\infty} \begin{pmatrix} a_{trs} \\ b_{trs} \end{pmatrix} z^s u^r v^t.$$

Terms with non-negative powers of z are holomorphic on U , so we can restrict to $s \leq -1$ and stay in the same cohomology class. Now on V ,

$$T\sigma = \begin{pmatrix} z^j a + pb \\ \sum_t \sum_r \sum_{s=-\infty}^{-1} z^{s-j} u^r v^t \end{pmatrix}.$$

Since $j \geq 0$, the second entry is holomorphic on V , and $T\sigma$ is cohomologous to

$$T\sigma \sim \begin{pmatrix} z^j a + pb \\ 0 \end{pmatrix}.$$

Going back to U , we find

$$T^{-1}T\sigma \sim \begin{pmatrix} a + z^{-j} p \sum_t \sum_r \sum_{s=-\infty}^{-1} b_{trs} z^s u^r v^t \\ 0 \end{pmatrix}.$$

Since no power of z in p is greater than $j-1$, we can relabel the coefficients and write

$$T^{-1}T\sigma \sim \begin{pmatrix} \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=-\infty}^{-1} a'_{trs} z^s u^r v^t \\ 0 \end{pmatrix}.$$

Going to V one last time, we find that the terms in

$$z^j \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=-\infty}^{-1} a'_{trs} z^s u^r v^t$$

are holomorphic on V and can be discarded if

$$s+j \leq \begin{cases} r+t & \text{on } W_1, \\ 2r & \text{on } W_2, \text{ and} \\ 3r-t & \text{on } W_3. \end{cases}$$

This constrains the exponents as follows:

$$C - j + 1 \leq s \leq -1, \text{ where } C := r + t, 2r, 3r - t \text{ respectively.}$$

So $C \leq j - 2$, and together with $r, t \geq 0$, the result follows. $\square$

3.3 Algebraicity and filtrability

In Chapter 2 we made use of the fact that bundles on the surfaces Z_k were algebraically filtrable, which is a consequence of the ampleness of the conormal bundle of the compact line inside the total space. We can apply the exact same reasoning to derive similar results for the spaces W_1 and W_2 and to see why W_3 does not possess these properties.

Theorem 3.10. *Every holomorphic vector bundle on $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ is filtrable and algebraic.*

Proof. This is a direct application of Theorem 2.2. $\square$

Theorem 3.11. *Let Z be the zero section of $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$. Fix an integer $r \geq 1$ and a holomorphic rank- r vector bundle E on $\hat{Z}$. Let $a_1 \geq \dots \geq a_r$ be the splitting type of $E|_Z$. Then there exist vector bundles F_i on $\hat{Z}$, $0 \leq i \leq r$, such that $F_r := E$, $F_1 := L_{a_1}$, $F_0 := \{0\}$ and $F_i|_Z$ has rank i and splitting type $a_1 \geq \dots \geq a_i$, and such that there are $r - 1$ exact sequences on $\hat{Z}$ (for $2 \leq i \leq r$)*

$$0 \longrightarrow L_{a_i} \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow 0, \quad (3.9)$$

where $L_{a_i} \cong \mathcal{O}(a_i)$.

Proof. The result is obvious if $r = 1$. Hence we may assume $r \geq 2$ and that the result is true for all vector bundles with rank at most $r - 1$. By assumption there is an injective map $j: \mathcal{O}_Z(a_r) \rightarrow E|_Z$ on Z such that $\text{coker}(j)$ is a rank- $(r - 1)$ vector bundle on Z with splitting type $a_1 \geq \dots \geq a_{r-1}$. The map j gives a nowhere-zero section s of $E(-a_r)|_Z$. Let us show that this section extends over a neighbourhood of Z : There is an exact sequence

$$0 \longrightarrow S^t(N_{Z,W_2}^*) \longrightarrow \mathcal{O}_Z^{(t+1)} \longrightarrow \mathcal{O}_Z^{(t)} \longrightarrow 0, \quad (3.10)$$

where $S^t(N_{Z,W_2}^*)$ is the t^{th} symmetric power of the conormal sheaf of Z in W_2 . In this case, we have $N_{Z,W_2} \cong \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z$, therefore,

$$S^t(N_{Z,W_2}^*) \cong \bigoplus_{k=0}^t \mathcal{O}_Z(2k).$$

After tensoring by the bundle $E(-a_r)$, the exact sequence (3.10) becomes

$$0 \longrightarrow E(-a_r) \otimes \left(\bigoplus_{k=0}^t \mathcal{O}_Z(2k) \right) \longrightarrow E(-a_r) \otimes \mathcal{O}_Z^{(t+1)} \longrightarrow E(-a_r) \otimes \mathcal{O}_Z^{(t)} \longrightarrow 0,$$

thus inducing the long cohomology sequence

$$\dots \longrightarrow H^0(Z; E(-a_r) \otimes \mathcal{O}_Z^{(t+1)}) \longrightarrow H^0(Z; E(-a_r) \otimes \mathcal{O}_Z^{(t)}) \longrightarrow \bigoplus_{k=0}^t H^1(Z; E(-a_r + 2k)) \longrightarrow \dots.$$

Note that $H^0(Z; E(-a_r) \otimes \mathcal{O}_Z^{(t)})$ is the space of global sections of $E(-a_r)$ on the t^{th} formal neighbourhood of Z in W_2 ; moreover, the obstruction to extending a section from the t^{th}

formal neighbourhood to the $(t+1)^{\text{st}}$ one lives in

$$\bigoplus_{k=0}^t H^1(Z; E(-a_r + 2k)).$$

However, since $E(-a_r)$ is a bundle of degree $\sum_{i=1}^{r-1} (a_i - a_r) \geq 0$, $E(-a_r + 2k)$ is of non-negative degree for $0 \leq k \leq t$, and thus all the cohomology groups $H^1(Z; E(-a_r + 2k))$ vanish for $0 \leq k \leq t$. Thus any section of $E(-a_r)$ on the t^{th} formal neighbourhood extends to the $(t+1)^{\text{st}}$. Hence, by Grothendieck's existence theorem ([Gro61, 5.1.4]), the section s extends to an actual neighbourhood of Z in W_2 , and consequently there is an exact sequence on W_2 of the form

$$0 \longrightarrow L_{a_r} \longrightarrow E \longrightarrow F_{r-1} \longrightarrow 0.$$

□

Corollary 3.12. *Every algebraic vector bundle on $W_2 = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1})$ is filtrable.*

Proof. Every such bundle is already determined on a finite infinitesimal neighbourhood of Z . □

Unlike in the case of W_1 , however, there are non-algebraic bundles on $\widehat{Z}$. It follows from Peternell's Existence theorem, though, that all such bundles do in fact extend to an analytic neighbourhood of Z .

3.4 The endomorphism bundle

If the bundle E is given by transition matrix $T = \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$, then the bundle $\text{End}(E) = E \otimes E^*$ is given by the transition matrix $T \otimes T^T$. After a convenient change of coordinates given by

$$P := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = P^{-1},$$

we will express the transition matrix of $E \otimes E^*$ as

$$S := P(T \otimes T^T)P = \begin{pmatrix} 1 & z^j p & z^{-j} p & p^2 \\ 0 & z^{2j} & 0 & z^j p \\ 0 & 0 & z^{-2j} & z^{-j} p \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ so } S^{-1} = \begin{pmatrix} 1 & -z^{-j} p & -z^j p & p^2 \\ 0 & z^{-2j} & 0 & -z^{-j} p \\ 0 & 0 & z^{2j} & -z^j p \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We are interested in $H^i(Z_k; \mathcal{E}nd E)$ for $i = 0, 1$. Like before, H^0 is the space of sections $(a, b, c, d) \in \Gamma(U; \mathcal{E}nd E) =: \Gamma_U$ such that $S(a, b, c, d) \in \Gamma_V$, while H^1 is the space of sections $\Gamma_{U \cap V}$ modulo $\Gamma_U \oplus \Gamma_V$.

A typical component of a section on U is given by $a(z, u) = \sum_{r,s \geq 0} a_{rs} z^s u^r$, and a section on $U \cap V$ is given by $a(z, u) = \sum_{r \geq 0} \sum_{s=-\infty}^{\infty} a_{rs} z^s u^r$. We have

$$S \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a + z^j p b + z^{-j} p c + p^2 d \\ z^{2j} b + z^j p d \\ z^{-2j} c + z^{-j} p d \\ d \end{pmatrix}.$$

We use this explicit form in the sequel to compute numerical invariants of the endomorphism bundle $\mathcal{E}nd(E)$.

3.5 Numerical invariants

By contrast to the case of bundles on surfaces, it is rather more involved to find numerical invariants of bundles on our threefolds W_i , $i = 1, 2, 3$. First, only on W_1 is the zero section Z contractible, so the concepts of width and height only make sense on W_1 , but not on W_2 or W_3 . Moreover, for codimensional reasons the width always vanishes on W_1 (see [BGK09, Lemma 5.2]). In this section we define several new numerical invariants that contain geometric information and provide a way of partitioning the moduli. These new numbers are “partial” invariants arising from restricting to a subspace, and invariants of the endomorphism bundle.

The spaces W_i contain two distinguished subsurfaces D_1 and D_2 (in fact degree-1 Cartier divisors), which are given in our canonical coordinates by the equations $D_1 \cap U = \{v = 0\}$ and $D_2 \cap U = \{u = 0\}$ on the U -chart and by $D_1 \cap V = \{z^i v = 0\}$ and $D_2 = \{z^{2-i} u = 0\}$ on the V -chart. By restricting to these surfaces, we define the following *partial invariants*:

Definition 3.13.

$$\begin{aligned} w'(E) &= w(E|_{D_1}) & w''(E) &= w(E|_{D_2}) \\ h'(E) &= h(E|_{D_1}) & h''(E) &= h(E|_{D_2}) \end{aligned} \quad (3.11)$$

Note that on W_1 both subspaces D_1 and D_2 are isomorphic to $Z_1 = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1))$. On W_2 , we have $D_2 \cong Z_2 = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2))$, but $D_1 \cong \mathbb{C}^2$, so we will only consider the restriction of bundles to D_2 . (In fact, we have an entire families of divisors from the pencils spanned by D_1 and D_2 . We will see this again later when we look at examples of moduli.)

Next we examine the endomorphism bundle $\mathcal{E}nd(E)$, which behaves very differently on W_1 and W_2 . The space W_1 is very similar to the surface Z_1 in many ways. On W_1 , the first cohomology group of $\mathcal{E}nd(E)$ is finite-dimensional, so its dimension is an invariant. We define:

$$\begin{aligned} h^1(E) &:= h^1(W_1; \mathcal{E}nd E) \\ \Delta_1 &:= h^1(W_1; \mathcal{E}nd(E_{\text{split}})) - h^1(W_1; \mathcal{E}nd(E)) \end{aligned} \quad (3.12)$$

The zeroth cohomology group of $\mathcal{E}nd E$ is infinite-dimensional, and we employ the same strategy as on Z_k : The infinitesimal neighbourhoods $Z^{(m)}$ are projective schemes, so the restrictions of $\mathcal{E}nd(E)$ to them have finite-dimensional cohomologies. Also, we can compute those dimensions for the endomorphism bundle of the split bundle of the same splitting type as E (denoted by E_{split}) and compare them. As m increases, this difference is eventually constant, and this gives our second invariant:

$$\Delta_0 := \lim_{n \rightarrow \infty} \left(h^0(W_1; \mathcal{E}nd(E_{\text{split}})|_{\ell^{(m)}}) - h^0(W_1; \mathcal{E}nd(E)|_{\ell^{(m)}}) \right) \quad (3.13)$$

The numbers we have just defined are not independent and satisfy several relations.

Proposition 3.14. *For all rank-2 bundles E on $X = Z_k, W_1, W_2$,*

$$h^1(X; \mathcal{E}nd E|_{\ell^{(m)}}) - h^0(X; \mathcal{E}nd E|_{\ell^{(m)}}) = h^1(X; \mathcal{E}nd(E_{\text{split}})|_{\ell^{(m)}}) - h^0(X; \mathcal{E}nd(E_{\text{split}})|_{\ell^{(m)}}).$$

Proof. We can express the statement in terms of the Hilbert polynomial

$$\phi_{\mathcal{F}^{(m)}}(n) := \chi(\mathcal{F}^{(m)}(n)) = h^1(X; \mathcal{F}|_{\ell^{(m)}}(n)) - h^0(X; \mathcal{F}|_{\ell^{(m)}}(n))$$

for any coherent sheaf $\mathcal{F}$ on X ; then the statement is

$$\phi_{\mathcal{E}nd E^{(m)}}(0) = \phi_{\mathcal{E}nd E_{\text{split}}^{(m)}}(0).$$

But in fact we have $\phi_{\mathcal{E}nd E^{(m)}}(n) = \phi_{E_{\text{split}}^{(m)}}(n)$ for any n and m , since the Hilbert polynomial is additive for short exact sequences of coherent sheaves over projective schemes (see Proposition 3.16 below), and $E^{(m)}$ is the extension

$$0 \longrightarrow \mathcal{O}_{\ell^{(m)}}(-j) \longrightarrow E^{(m)} \longrightarrow \mathcal{O}_{\ell^{(m)}}(j) \longrightarrow 0.$$

Thus both $\mathcal{E}nd E$ and $\mathcal{E}nd E_{\text{split}}$ have a filtration

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow F_4 \longrightarrow 0,$$

where $F_1 \cong \mathcal{O}(-2j)$, $F_2/F_1 \cong \mathcal{O}$, $F_3/F_2 \cong \mathcal{O}$, $F_4/F_3 \cong \mathcal{O}(2j)$, and thus their Hilbert polynomials coincide. $\square$

Corollary 3.15. *For all rank-2 bundles E on $X = Z_k, W_1$, we have $\Delta_0(E) = \Delta_1(E)$, or equivalently $\Delta_0(E) + h^1(E) = h^1(X; \mathcal{E}nd(E_{\text{split}}))$.*

Proof. This follows from Proposition 3.14 by fact that $H^1(X; \mathcal{E}nd E)$ is already determined on a finite neighbourhood $\ell^{(m)} \subset X$ and by unravelling the definitions of Δ_0 and Δ_1 . $\square$

Proposition 3.16. *Suppose $(X, \mathcal{O}_X)$ is a projective scheme with a fixed, ample twisting sheaf $\mathcal{O}_X(1)$, and $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ a short exact sequence of coherent $\mathcal{O}_X$ -sheaves. Then the Hilbert polynomials satisfy $\phi_{\mathcal{F}} = \phi_{\mathcal{F}'} + \phi_{\mathcal{F}''}$.*

Proof. By definition of ampleness, there exists a number n such that $\mathcal{E}(n)$ is generated by global sections for $\mathcal{E} = \mathcal{F}', \mathcal{F}, \mathcal{F}''$, and thus $H^i(X; \mathcal{E}) = 0$ for $i > 0$, and thus $\phi_{\mathcal{E}}(m) = h^0(X; \mathcal{E}(m))$ for $m \geq n$. The short exact sequence of the hypothesis induces a long exact sequence

$$0 \longrightarrow H^0(X; \mathcal{F}'(m)) \longrightarrow H^0(X; \mathcal{F}(m)) \longrightarrow H^0(X; \mathcal{F}''(m)) \longrightarrow \underline{H^1(X; \mathcal{F}'(m))},$$

and the result follows. $\square$

Lemma 3.17. *Let E be an extension of type 2.1 with splitting type j on either Z_k or W_1 . Then the Hilbert polynomial of $E|_{\ell^m}$ is*

$$n \mapsto \chi(E^{(m)}(n)) := \sum_i (-1)^i h^i(\ell^{(m)}; E(n)) = \begin{cases} (m+1)(km+2+2n) & \text{on } Z_k, \text{ and} \\ \frac{1}{3}(m+2)(m+1)(2m+3n+3) & \text{on } W_1, \end{cases}$$

independent of the extension class, and independent of the splitting type j . Similarly, the Hilbert polynomial of the endomorphism bundle $\mathcal{E}nd E|_{\ell^{(m)}}$ is $2\chi(E^{(m)}(n))$.

Proof. It follows from the proof of Proposition 3.14 that the Hilbert polynomials in question are determined by the Hilbert polynomial of the line bundles $\mathcal{O}_{\ell^{(m)}}(p)$ for all p . Since $\mathcal{O}_{\ell^{(m)}}(1)$ is ample, the higher cohomology of $\mathcal{O}_{\ell^{(m)}}(p)$ vanishes for sufficiently large p . (We can verify this by direct computation.)

Being a polynomial, the Hilbert polynomial is determined by finitely many values, so it suffices to compute $\phi_{\mathcal{E}nd E^{(m)}}(n) = h^0(\ell^{(m)}; \mathcal{O}_{\ell^{(m)}}(p))$ for large p . By the additivity of the Hilbert polynomial (Proposition 3.16) and the fact that E and $\mathcal{E}nd E$ have filtrations by line

bundles (as given in the proof of Proposition 3.14) which restrict to filtrations on every infinitesimal neighbourhood $\ell^{(m)}$, we compute:

$$\begin{aligned}\phi_{E^{(m)}}(n) &= \phi_{\mathcal{O}_{\ell^{(m)}}(-j)}(n) + \phi_{\mathcal{O}_{\ell^{(m)}}(j)}(n) \\ \phi_{\mathcal{E}nd E^{(m)}}(n) &= \phi_{\mathcal{O}_{\ell^{(m)}}(-2j)}(n) + 2\phi_{\mathcal{O}_{\ell^{(m)}}}(n) + \phi_{\mathcal{O}_{\ell^{(m)}}(2j)}(n)\end{aligned}$$

We conclude this proof by computing $H^0(\ell^{(m)}; \mathcal{O}(p))$. Now we have to consider the spaces Z_k and W_1 separately.

On $\ell^{(m)} \subset Z_k$, a section $a \in \mathcal{O}(p)(U)$ is $a(z, u) = \sum_{r=0}^m \sum_{s=0}^{\infty} a_{rs} z^s u^r$ such that $\sum_{r,s} a_{rs} z^{s-p} u^r$ is holomorphic in $(z^{-1}, z^k u)$, i.e. $s - p \leq kr$. Thus

$$a(z, u) = \sum_{r=0}^m \sum_{s=0}^{kr+p} a_{rs} z^s u^r,$$

which has $\frac{1}{2}(m+1)(km+2+2p) =: \phi_{\mathcal{O}}(p)$ coefficients.

On $\ell^{(m)} \subset W_1$, a section $a \in \mathcal{O}(p)(U)$ is $a(z, u, v) = \sum_{t=0}^m \sum_{r=0}^{m-t} \sum_{s=0}^{\infty} a_{trs} z^s u^r v^t$ such that $\sum_{t,r,s} a_{trs} z^{s-p} u^r v^t$ is holomorphic in (z^{-1}, zu, zv) , i.e. $s - p \leq r + t$. Thus

$$a(z, u, v) = \sum_{t=0}^m \sum_{r=0}^{m-t} \sum_{s=0}^{r+t+p} a_{trs} z^s u^r v^t,$$

which has $\frac{1}{6}(m+2)(m+1)(2m+3p+3) =: \phi_{\mathcal{O}}(p)$ coefficients.

Putting it all together, we have

$$\begin{aligned}\phi_{E^{(m)}}(n) &= \phi_{\mathcal{O}}(-j+n) + \phi_{\mathcal{O}}(j+n), \\ \phi_{\mathcal{E}nd E^{(m)}}(n) &= \phi_{\mathcal{O}}(-2j+n) + 2\phi_{\mathcal{O}}(n) + \phi_{\mathcal{O}}(2j+n),\end{aligned}$$

which gives the desired functions. □

On W_2 the situation is different. Since $W_2 \cong Z_2 \times \mathbb{C}$, the Künneth formula shows that both the zeroth and the first cohomology groups of $\mathcal{E}nd(E)$ are infinite-dimensional. But when we use the same strategy and compare the dimensions of the cohomology groups of the restrictions to the m^{th} infinitesimal neighbourhood of $\mathcal{E}nd(E)$ and $\mathcal{E}nd(E_{\text{split}})$, we find that their difference increases linearly in m .

In fact, more is true. Rearranging the equation of Proposition 3.14, we see that

$$h^0(X; \mathcal{E}nd(E_{\text{split}}|_X)) - h^0(X; \mathcal{E}nd E|_X) = h^1(X; \mathcal{E}nd(E_{\text{split}})) - h^1(X; \mathcal{E}nd E|_X) = cm + d,$$

so we obtain two numbers, the slope c and the intercept d of the dimension difference function. (If we had made the same definition on W_1 , we would just get $c = 0$ and $d = \Delta_0 = \Delta_1$.)

Example values on W_1 are tabulated in Table 3.1 and on W_2 in Table 3.2, and we summarize the numerical invariants that we can compute on the spaces W_1 , W_2 and W_3 :

W_1	height, h' , h'' , w' , w'' , Δ_0 , Δ_1 , h^1
W_2	h'' , w'' , c , d
W_3	h'' , w''

Conjecture: On the surface Z_k , we have $w(E) + h(E) = \chi(E) = \frac{h^1 - \Delta_0 - j}{2} + \frac{j}{k}$.

j	p	Δ_0	Δ_1	h^1	(w', h')	(w'', h'')	height
3	0	0	0	35	(6, 3)	(6, 3)	4
3	u	15	15	20	(1, 2)	(6, 3)	3
3	zu	15	15	20	(1, 2)	(6, 3)	3
3	$v + u$	15	15	20	(1, 2)	(1, 2)	3
3	$v + zu$	18	18	17	(1, 2)	(1, 2)	2
3	z^2u	10	10	25	(3, 2)	(6, 3)	3
3	$z^{-1}u$	10	10	25	(3, 2)	(6, 3)	3
3	$z^{-1}u + u$	15	15	20	(1, 2)	(6, 3)	3
3	$z^{-1}u + zu$	15	15	20	(1, 2)	(6, 3)	3
3	$z^{-1}u + z^2u$	15	15	20	(1, 2)	(6, 3)	3
3	$z^{-1}u + z^2v$	16	16	19	(3, 2)	(3, 2)	2
3	$z^{-1}v + z^2u$	16	16	19	(3, 2)	(3, 2)	2
3	$z^{-1}u + z^{-1}v$	10	10	25	(3, 2)	(3, 2)	3

Table 3.1: Example data on W_1 . Observe that h^1 (or Δ_1) is a finer measure of genericity than the height.

j	p	c_0	d_0	c_1	d_1	H	(w'', h'')
3	0	0	0	0	0	0	(2, 2)
3	u	2	0	2	9	0	(1, 2)
3	zu	2	1	2	9	2	(0, 2)
3	z^2u	2	1	2	9	0	(1, 2)
3	$z^2u + u$	2	1	2	9	2	(0, 2)
3	$z^{-2}v$	3	-3	3	12	6	(2, 2)
3	z^2v	3	0	3	12	3	(2, 2)
3	$z^2v + u$	3	0	3	13	4	(1, 2)
3	$z^{-1}v$	4	-2	4	16	3	(2, 2)
3	zv	4	0	4	16	4	(2, 2)
3	$u + zv$	4	0	4	17	5	(1, 2)
3	$u + z^{-1}v$	4	-2	4	16	6	(1, 2)
3	$zu + z^{-1}v$	4	-2	4	16	6	(0, 2)
3	v	5	-1	5	20	6	(2, 2)
3	$u + v$	5	-1	5	20	6	(1, 2)
3	$zu + v$	5	-1	5	20	6	(0, 2)
3	$zv + v$	5	-1	5	20	6	(2, 2)

Table 3.2: Example data on W_2 .

Space	Split bundle E_j	Generic bundle G_j
$Z_k, k = 2n$	$\frac{-z(z^{n+1} + z^n + z + 1)}{(z-1)^2(z^k - 1)}$	$\frac{z^{k+2} - z^3 - z^2 - z}{(z-1)^2(z^k - 1)}$
$Z_k, k = 2n + 1$	$\frac{-z(2z^{n+1} + z + 1)}{(z-1)^2(z^k - 1)}$	
W_1	$\frac{z(z+6) + 1}{(z-1)^4}$	$\frac{z(-z^2 + 2z + 1)}{(z-1)^4}$

Table 3.3: Generating functions for $h^1(\mathcal{E}nd E)$ on various spaces for the split and the generic bundle of splitting type j (data for G_j only valid for $j \geq k$); the value is the j^{th} coefficient in the Taylor series.

New invariant? It appears that on W_2 we always have $c_0 = c_1$. Therefore we can define a new number

$$H(E) := \lim_{m \rightarrow \infty} \left(h^1(W_2; \mathcal{E}nd E_{\text{split}}|_{\ell^{(m)}}) - h^1(W_2; \mathcal{E}nd E|_{\ell^{(m)}}) \right) - \left(h^0(W_2; \mathcal{E}nd E_{\text{split}}|_{\ell^{(m)}}) - h^0(W_2; \mathcal{E}nd E|_{\ell^{(m)}}) \right) = \lim_m (\phi_{E_{\text{split}}, m}(0) - \phi_{E, m}(0)),$$

where we used the notation “ H ” to indicates the relation to the Hilbert polynomials.

Remark 3.18. The corresponding quantity H on the space W_1 is always zero, which is equivalent to the identity $\Delta_0 = \Delta_1$ proved above.

Direct computation lead us to discover a compact expression for the number $h^1(\mathcal{E}nd E)$ on the spaces Z_k and W_1 , where $\mathcal{E}$ is either the generic or the split bundle of splitting type j (with $j \geq k$ on Z_k).

Definition 3.19. A power series of the form $g(z) = \sum_{j=0}^{\infty} a_j z^j$ is called a *generating function* for the sequence $(a_j)_{j=0}^{\infty}$. Hence, $a_j = \frac{1}{j!} \frac{d^j g}{dz^j} \Big|_{z=0}$.

In Table 3.3 we present the generating functions for the series $a_j^{X,E} := h^1(\mathcal{E}nd E)$ on the spaces Z_k and W_1 for the generic and the split bundle of splitting type j . A few series for special values can be listed explicitly:

- For the split instanton bundle E_j , $j = kn$ on Z_k , $h^1(Z_k; \mathcal{E}nd E_j) = n(2nk + k - 2)$.
- For a generic instanton bundle G_j , $j = kn$ on Z_k , we have $h^1(Z_k; \mathcal{E}nd G_j) = n(nk + 2k - 2) - 1$ for $k \geq 2$ and $h^1(Z_1; \mathcal{E}nd G_j) = j^2$.
- On W_1 , we have for the split bundle E_j , $h^1(W_1; \mathcal{E}nd E_j) = (4j^3 - j)/3$. This equals the number of coefficients in the generalised extension class $\tilde{p}$ in Remark 3.3.

3.6 Moduli

If we restrict our attention to the moduli space of bundles that are extensions of line bundles and do not split on the first infinitesimal neighbourhood, we can apply Equation 1.1 to all three of the spaces W_1 , W_2 and W_3 .

Proposition 3.20. *The part of $\mathfrak{M}(W_i; j)$, $i = 1, 2, 3$, which consists of extensions that do not split on the first infinitesimal neighbourhood $Z^{(1)}$ is smooth and has dimension $\gamma_1 - 1$, i.e.*

$$\dim(\mathfrak{M}(j)|_{\text{generic}}) = 4j - 5 \quad \text{for } j \geq 2.$$

Further, the generic set of $\mathfrak{M}(1)$ is empty on W_1 , a point on W_2 and one-dimensional on W_3 .

Proof. Let $\mathbb{C}^{\gamma_1}$ be the space of coefficients p_{10s} and p_{01s} of p . By Proposition 3.5 the only isomorphisms on $\ell^{(1)}$ are scaling, and thus $\mathfrak{M}(j)|_{\text{generic}}$ is obtained by projectivising the open subset of generic coefficients of the affine space $\mathbb{C}^{\gamma_1}$.

Now we just compute γ_1 directly as in Equation (1.1): We have $F = \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$, so

$$\mathcal{E}nd F \cong \mathcal{O}_{\mathbb{P}^1}(-2j) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2j) \cong (\mathcal{E}nd F)^{\vee}.$$

Also,

$$N_{\ell, X_1} = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \quad N_{\ell, X_2} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1} \quad N_{\ell, X_3} = \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \quad ,$$

and by Serre duality,

$$\begin{aligned} \gamma_1 &= h^0(\ell; (\mathcal{E}nd F \otimes N_{\ell, X_i}^*)^{\vee} \otimes \omega_{\mathbb{P}^1}) \\ &= h^0(\ell; (\mathcal{E}nd F)^{\vee} \otimes N_{\ell, X_i} \otimes \omega_{\mathbb{P}^1}) \\ &= 4(j-1) \quad \text{for all } X_i \text{ and } j \geq 2. \end{aligned}$$

The results for $j = 1$ follow from the same computation. $\square$

Remark 3.21. The result of Proposition 3.20 is sharp among the spaces W_i in the sense that it is only true for $i = 1, 2, 3$; for any $i > 3$ the value of γ_1 is greater than $4(j-1)$.

3.6.1 Bundles on W_1

The space $W_1 := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, also known as the space of the simple flop, was already studied in passing in [BGK09]. Here we present a more detailed treatment, which also illustrates the connection between the parameter space of bundles and the moduli of isomorphism classes of extensions. We give an explicit description of $\mathfrak{M}(W_1; j)$ for small j :

Proposition 3.22 (Moduli on W_1 for $j = 0, 1$).

Let E be a bundle on W_1 of splitting type $j = 0$. Then $E \cong \mathcal{O}^{\oplus 2}$, i.e. E is trivial.

Let E be a bundle on W_1 of splitting type $j = 1$. Then $E \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$, i.e. E splits.

Proof. This follows immediately from Proposition 3.1: On W_1 for $j = 0$ or $j = 1$ we can always write the extension as $p = 0$, so every rank-2 bundle of this splitting type is the split bundle. $\square$

Remark 3.23. Thus there are no “generic” bundles of splitting type 0 or 1 on W_1 , i.e. no bundles that do not split on $Z^{(1)}$.

Proposition 3.24 (Moduli on W_1 for $j = 2$). *Let E be a rank-2 vector bundle on W_1 given by the transition matrix*

$$\begin{pmatrix} z^2 & p(z, u, v) \\ 0 & z^{-2} \end{pmatrix}.$$

The space of isomorphism classes of such bundles is the set

$$\{(p_{010}, p_{011}, p_{100}, p_{101}), p_{021}, p_{201}, p_{111}\} \subseteq \mathbb{C}^7$$

modulo a set of relations presented at the end of the proof.

Proof. By Proposition 3.1, a bundle on W_1 with splitting type $j = 2$ is determined by its extension class

$$p(z, u, v) = (p_{010} + p_{011}z)u + (p_{100} + p_{101}z)v + p_{021}zu^2 + p_{201}zv^2 + p_{111}zuv.$$

Suppose we have two such bundles E and E' , given respectively by extension classes p and q . We write $p|_{Z^{(1)}}$ for the restriction to the first formal neighbourhood, i.e. to terms that have total degree ≤ 1 in u, v ; so

$$p|_{\ell_1}(z, u, v) = (p_{010} + p_{011}z)u + (p_{100} + p_{101}z)v,$$

and similarly for q .

If $E \cong E'$, then by Proposition 3.5, $p|_{Z^{(1)}} = \lambda q|_{Z^{(1)}}$ for some $\lambda \in \mathbb{C}^\times$, and by rescaling we may assume that $\lambda = 1$. (We return to this point at the end of the proof.) Now we identify sufficient conditions for an isomorphism: The isomorphism, if it exists, can be written as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z^2 & p \\ 0 & z^{-2} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} z^{-2} & -q \\ 0 & z^2 \end{pmatrix} \\ = \begin{pmatrix} A + z^{-2}pC & z^4B + z^2(pD - qA) - pqC \\ z^{-4}C & D - z^{-2}qC \end{pmatrix},$$

where A, B, C, D are holomorphic on $U \cap \widehat{Z}$ and $\alpha, \beta, \gamma, \delta$ on $V \cap \widehat{Z}$, i.e. power series in (z, u, v) or (z^{-1}, zu, zv) , respectively. As usual we write $A = \sum a_{trs} z^s u^r v^t$ etc.

First, the condition that the $(2, 1)$ -entry be holomorphic in (z^{-1}, zu, zv) implies

$$C = \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{r+t+4} c_{trs} z^s u^r v^t.$$

Next, the condition of holomorphy of the diagonal entries yields the following relations for the terms of the power series A and D :

$$\begin{aligned} - \begin{pmatrix} a_{012} \\ a_{013} \\ a_{014} \\ a_{015} \end{pmatrix} &= \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{001} \\ c_{002} \\ c_{003} \\ c_{004} \end{pmatrix} \\ - \begin{pmatrix} a_{102} \\ a_{103} \\ a_{104} \\ a_{105} \end{pmatrix} &= \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{001} \\ c_{002} \\ c_{003} \\ c_{004} \end{pmatrix} \\ - \begin{pmatrix} a_{023} \\ a_{024} \\ a_{025} \\ a_{026} \end{pmatrix} &= \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{012} \\ c_{013} \\ c_{014} \\ c_{015} \end{pmatrix} + \begin{pmatrix} p_{021} & 0 & 0 & 0 \\ 0 & p_{021} & 0 & 0 \\ 0 & 0 & p_{021} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix} \\ - \begin{pmatrix} a_{203} \\ a_{204} \\ a_{205} \\ a_{206} \end{pmatrix} &= \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{102} \\ c_{103} \\ c_{104} \\ c_{105} \end{pmatrix} + \begin{pmatrix} p_{201} & 0 & 0 & 0 \\ 0 & p_{201} & 0 & 0 \\ 0 & 0 & p_{201} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix} \\ - \begin{pmatrix} a_{113} \\ a_{114} \\ a_{115} \\ a_{116} \end{pmatrix} &= \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{012} \\ c_{013} \\ c_{014} \\ c_{015} \end{pmatrix} + \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{102} \\ c_{103} \\ c_{104} \\ c_{105} \end{pmatrix} + \begin{pmatrix} p_{111} & 0 & 0 & 0 \\ 0 & p_{111} & 0 & 0 \\ 0 & 0 & p_{111} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} d_{012} \\ d_{013} \\ d_{014} \\ d_{015} \end{pmatrix} &= \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{001} \\ c_{002} \\ c_{003} \\ c_{004} \end{pmatrix} \\
\begin{pmatrix} d_{102} \\ d_{103} \\ d_{104} \\ d_{105} \end{pmatrix} &= \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{001} \\ c_{002} \\ c_{003} \\ c_{004} \end{pmatrix} \\
\begin{pmatrix} d_{023} \\ d_{024} \\ d_{025} \\ d_{026} \end{pmatrix} &= \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{012} \\ c_{013} \\ c_{014} \\ c_{015} \end{pmatrix} + \begin{pmatrix} p_{021} & 0 & 0 & 0 \\ 0 & p_{021} & 0 & 0 \\ 0 & 0 & p_{021} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix} \\
\begin{pmatrix} d_{203} \\ d_{204} \\ d_{205} \\ d_{206} \end{pmatrix} &= \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{102} \\ c_{103} \\ c_{104} \\ c_{105} \end{pmatrix} + \begin{pmatrix} p_{201} & 0 & 0 & 0 \\ 0 & p_{201} & 0 & 0 \\ 0 & 0 & p_{201} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix} \\
\begin{pmatrix} d_{113} \\ d_{114} \\ d_{115} \\ d_{116} \end{pmatrix} &= \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{012} \\ c_{013} \\ c_{014} \\ c_{015} \end{pmatrix} + \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{102} \\ c_{103} \\ c_{104} \\ c_{105} \end{pmatrix} + \begin{pmatrix} p_{111} & 0 & 0 & 0 \\ 0 & p_{111} & 0 & 0 \\ 0 & 0 & p_{111} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix}
\end{aligned}$$

Lastly, the (1, 2)-entry,

$$z^4 B + z^2(pD - qA) - pqC,$$

has to be made holomorphic in (z^{-1}, zu, zv) . Since $z^4 B$ can be chosen to cancel higher terms, we only need to consider those terms $z^s u^r v^t$ with $s > r + t$ and $s < 4$ in the expression $z^2(pD - qA) - pqC$.

Recall that A and D are of the form $1 + u(\dots) + v(\dots)$ and that we have already set $p|_{Z^{(1)}} = q|_{Z^{(1)}}$. Thus the coefficients of u and v are zero. It remains to find the coefficients of the terms u^2 , v^2 and uv . From this we get three equations:

$$\begin{aligned}
0 &= (p_{021} - q_{021}) - c_{003}p_{010}^2 - 2c_{002}p_{010}p_{011} - c_{001}p_{011}^2 \\
&\quad + p_{010}(d_{011} - a_{011}) + p_{011}(d_{010} - a_{010})
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
0 &= (p_{201} - q_{201}) - c_{003}p_{100}^2 - 2c_{002}p_{100}p_{101} - c_{001}p_{101}^2 \\
&\quad + p_{100}(d_{101} - a_{101}) + p_{101}(d_{100} - a_{100})
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
0 &= (p_{111} - q_{111}) - 2c_{003}p_{010}p_{101} - 2c_{002}(p_{010}p_{101} + p_{011}p_{100}) \\
&\quad - 2c_{001}p_{011}p_{101} + p_{100}(d_{011} - a_{011}) + p_{010}(d_{101} - a_{101}) \\
&\quad + p_{101}(d_{010} - a_{010}) + p_{011}(d_{100} - a_{100})
\end{aligned} \tag{3.16}$$

Finally we describe the moduli of extensions of splitting type $j = 2$ as the space of coefficients

$$X := \{(p_{010}, p_{011}, p_{100}, p_{101}), p_{021}, p_{201}, p_{111}\}$$

modulo relations that we infer from the above equations:

1. If $(p_{010}, p_{011}) \neq (0, 0)$, Equations (3.14) and (3.16) can be solved for any $p_{021}, p_{201}, p_{111}$.
2. If $(p_{100}, p_{101}) \neq (0, 0)$, Equations (3.15) and (3.16) can be solved for any $p_{021}, p_{201}, p_{111}$.
3. If all first-order coefficients vanish, the three equations imply that all the second-order coefficients are equal.

Thus the moduli consists of the following sets:

1. The generic set $S := \{p_{010}, p_{011}, p_{100}, p_{101}\}/\mathbb{C}^\times$, where $(p_{010}, p_{011}) \neq (0, 0)$ and $(p_{100}, p_{101}) \neq (0, 0)$,

2. the set $T_1 := \{(0, 0, p_{100}, p_{101}), p_{021}\} / \mathbb{C}^\times$, where $p_{021} \in \mathbb{C}$ and $(p_{100}, p_{101}) \neq (0, 0)$,
3. the set $T_2 := \{(p_{010}, p_{011}, 0, 0), p_{201}\} / \mathbb{C}^\times$, where $p_{201} \in \mathbb{C}$ and $(p_{010}, p_{011}) \neq (0, 0)$,
4. the set $\{(0, 0, 0, 0), p_{021}, p_{201}, p_{111}\} / \mathbb{C}^\times$, where $(p_{021}, p_{201}, p_{111}) \neq (0, 0, 0)$, and
5. the split bundle (all $p_{trs} = 0$).

□

Conclusion: The moduli of extensions of splitting type $j = 2$ on W_1 is

$$\mathfrak{M}(W_1; 2) = S \sqcup T_1 \sqcup T_2 \sqcup \mathbb{C}P^2 \sqcup \{*\},$$

where the generic set is

$$S = \{[p_{010} : p_{011} : p_{100} : p_{101}] \in \mathbb{C}P^3 : (p_{010}, p_{011}) \neq (0, 0) \text{ and } (p_{100}, p_{101}) \neq (0, 0)\}.$$

Furthermore,

$$\begin{aligned} T_1 &= \{[p_{100} : p_{101} : p_{021}] \setminus [0 : 0 : 1]\} \cong \mathbb{C}P^2 \setminus \{*\} \text{ and} \\ T_2 &= \{[p_{010} : p_{011} : p_{201}] \setminus [0 : 0 : 1]\} \cong \mathbb{C}P^2 \setminus \{*\}. \end{aligned}$$

The generic set S has dimension $4 \cdot 2 - 5 = 3$, as in the dimension count Proposition 3.20 promised. Also, S is a proper subset of $\mathbb{C}^4$, since the bundles with $(p_{100}, p_{101}) = (0, 0)$ or $(p_{010}, p_{011}) = (0, 0)$ are *not* generic, and those form a closed subset. Note that the generalised extension class for $j = 2$ is

$$\begin{aligned} \tilde{p}(z, u, v) &= p_{0,0,-1}z^{-1} + p_{000} + p_{001}z + (p_{010} + p_{011}z)u \\ &\quad + (p_{100} + p_{101}z)v + p_{021}zu^2 + p_{201}zv^2 + p_{111}zuv, \end{aligned}$$

which has $\gamma = 10$ coefficients, in accord with Remark 3.3, the first three of which are deformations along Z into bundles of lower splitting type.

We can express this result compactly as follows: The moduli $\mathfrak{M}(W_1; 2)$ is the space of orbits in $\mathbb{C}^7$ of the action

$$\begin{pmatrix} p_{010} \\ p_{011} \\ p_{100} \\ p_{101} \\ p_{021} \\ p_{111} \\ p_{201} \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & 0 & 0 & \lambda & 0 & 0 \\ \alpha_3 & \alpha_4 & \beta_3 & \beta_4 & 0 & \lambda & 0 \\ 0 & 0 & \beta_1 & \beta_2 & 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} p_{010} \\ p_{011} \\ p_{100} \\ p_{101} \\ p_{021} \\ p_{111} \\ p_{201} \end{pmatrix},$$

where $\lambda \in \mathbb{C}^\times$ and $\alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, \dots, 4$. Note that the group G that acts is not reductive, and thus the quotient is not amenable to standard GIT techniques. Explicitly, G is given as the extension $0 \rightarrow (\mathbb{C}^\times, \times) \rightarrow G \rightarrow (\mathbb{C}, +)^8 \rightarrow 0$.

We see directly that the restriction of the action of G to the subspace $\mathbb{C}^4$ spanned by $\{p_{010}, p_{011}, p_{100}, p_{101}\}$ reduces to $\mathbb{C}^\times$, which acts faithfully, and this subspace is the largest subset whose quotient by the G -action is Hausdorff. The set of generic bundles that we identified above is a Zariski-open subset of this quotient.

The numerical invariants h, h', w', h'', w'' and h^1 help distinguish the different types of bundles, and they are tabulated in Table 3.4. The table gives also the additional numbers Δ_0

$p(z, u, v)$	cpt. of $\mathfrak{M}(W_1; 2)$	h	h'	w'	h''	w''	h^1	Δ_1	Δ_0
0	$\{*\}$	1	1	3	1	3	10	0	0
zuv	$\mathbb{C}P^2$	1	1	3	1	3	9	1	1
zu^2	$\mathbb{C}P^2$	1	1	2	1	3	9	1	1
v	T_1	1	1	3	1	1	7	3	3
u	T_2	1	1	1	1	3	7	3	3
$u + v$	S	1	1	1	1	1	7	3	3
$u + zv$	S	1	1	1	1	1	6	4	4

Table 3.4: Numerical invariants of several bundles on W_1 of splitting type $j = 2$.

and Δ_1 , but recall that they are determined by h^1 . While those numerical invariants are not quite sufficient to give $\mathfrak{M}(W_1; 2)$ a Hausdorff decomposition, it does suffice to identify the generic set S , which is the one where the sum of the h', w', h'', w'' is minimal; or alternatively where $h^1 \leq 7$.

Table 3.4 exhibits another phenomenon: The two bundles given by zu^2 and zuv are clearly in the same part of the moduli and related by a change of coordinates, yet the partial invariants w' and w'' differ; the “correct” value is given by $w(E_{\{u+v=0\}}) = 3$. To make the use of partial invariants general, we could devise a family version parametrised by $[\lambda : \mu] \in \mathbb{P}^1$ computing $w(E_{\{\lambda u + \mu v = 0\}})$. The number h^1 actually provides a finer invariant than needed, as the generic set consists of those bundles with $h^1 \leq 7$.

3.6.2 Bundles on W_2

The crucial difference between W_2 and W_1 is that $N_{Z, W_2}^* \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$ is not ample. We see both from the form of the canonical extension class in Proposition 3.1 and from the dimension count (1.1) that the parameter space for extensions $p(z, u, v) = \sum p_{trs} z^s u^r v^t$ is infinite-dimensional, and it is clear that there exist non-algebraic bundles, e.g. on the subspace $\text{Tot}(0 \oplus \mathcal{O}_{\mathbb{P}^1})$. Nonetheless, we saw in Theorem 3.11 that every bundle on W_2 is still filtrable.

This means that the moduli of all rank-2 bundles with vanishing first Chern class is still a union of moduli $\mathfrak{M}(W_2; j)$ of extensions of fixed splitting type j . Even though each $\mathfrak{M}(W_2; j)$ is now in some sense infinite, we can still attempt to describe it. We start with a few moduli $\mathfrak{M}(W_2; j)$ for small j . The case $j = 0$ is easy:

Proposition 3.25 (Moduli on W_2 for $j = 0$). *Let E be a bundle on Λ_2 of splitting type $j = 0$. Then $E \cong \mathcal{O}^{\oplus 2}$, i.e. E is trivial.*

Proof. This follows immediately from Proposition 3.1: On W_2 for $j = 0$ we can always write the extension as $p = 0$, so every rank-2 bundle of this splitting type is trivial. $\square$

For $j = 1$, Proposition 3.20 shows that there is only one single generic bundle. The full space $\mathfrak{M}(W_2; 1)$ can be described as follows. Substituting $j = 1$ into Proposition 3.1, we see that the polynomial p must be of the form $p(z, u, v) = \sum_{t=1}^{\infty} p_{t00} v^t$.

Proposition 3.26 (Moduli on W_2 for $j = 1$). *Let E_p, E_q be two bundles on W_2 of splitting type $j = 1$ determined by polynomials p, q , respectively. E_p and E_q are isomorphic if and only if one of the following conditions hold:*

- If $p \equiv 0 \equiv q$; or

- if $p_1 \neq 0 \neq q_1$; or
- if $p_i = 0 = q_i$ for $i \geq 1$ and $p_{i+1} = q_{i+1} \neq 0$.

Proof. We determine conditions for E_q and E_q to be isomorphic by rerunning the proof of Proposition 3.24: We need transformation matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z & p \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} z^{-1} & -q \\ 0 & z \end{pmatrix} \\ = \begin{pmatrix} A + z^{-1}pC & z^2B + z(pD - qA) - pqC \\ z^{-2}C & D - z^{-1}qC \end{pmatrix},$$

where A, B, C, D are holomorphic on $U \cap \hat{\ell}$ and $\alpha, \beta, \gamma, \delta$ on $V \cap \hat{\ell}$, i.e. formal power series in (z, u, v) or (z^{-1}, z^2u, v) , respectively.

First, the condition that the $(2, 1)$ -entry be holomorphic in (z^{-1}, z^2u, v) implies

$$C = \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{2r+2} c_{trs} z^s u^r v^t.$$

Second, we obtain another set of equations like those on page 30, which we omit here to avoid cluttering the presentation.

Finally, the $(1, 2)$ -entry

$$z^2B + z(pD - qA) - pqC$$

has to be made holomorphic in (z^{-1}, z^2u, v) . Since z^2B can be chosen to cancel higher terms, we only need to consider those terms $z^s u^r v^t$ with $s > 2r$ and $s < 2$ in the expression $z(pD - qA) - pqC$. This leaves only the terms $z v^t$ with $t \geq 1$. Moreover, by Proposition 3.5, $E \cong E'$ implies that we can scale q such that $p|_{\ell_1} = q|_{\ell_1}$. Thus in fact we only need to consider $t \geq 2$.

$$\begin{aligned} & z \left((p_1 v + p_2 v^2 + p_3 v^3 + \dots)(1 + u(\dots) + v(d_{100} + d_{101}z + d_{102}z^2 + \dots)) \right. \\ & \quad \left. - (p_1 v + q_2 v^2 + q_3 v^3 + \dots)(1 + u(\dots) + v(a_{100} + a_{101}z + a_{102}z^2 + \dots)) \right) \\ & \quad - (p_1 v + p_2 v^2 + p_3 v^3 + \dots)(p_1 v + q_2 v^2 + q_3 v^3 + \dots) \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{2r+2} c_{trs} z^s u^r v^t \end{aligned}$$

We obtain an infinite series of equations. The first few are:

$$\text{For } v^2 z: \quad p_1(d_{100} - a_{100}) + (p_2 - q_2) - p_1^2 c_{001} = 0.$$

$$\text{For } v^3 z: \quad p_1(d_{200} - a_{200}) + (p_2 d_{100} - q_2 a_{100}) + (p_3 - q_3) - p_1^2 c_{101} - (p_1 q_2 + p_2 q_1) c_{001} = 0.$$

$$\vdots \quad \quad \quad \vdots$$

From this we obtain the following infinite list of families of extensions:

- The split bundle, $p = 0$.
- One “generic” bundle $p = v$, isomorphic to all $p = p_1 v + \sum_{t \geq 2} p_t v^t$ for $p_1 \neq 0$.
- A family $p = p_2 v^2$ with $p_2 \in \mathbb{C}^\times$, each member being isomorphic to $p = p_2 v^2 + \sum_{t \geq 3} p_t v^t$.
- A family $p = p_3 v^3$ with $p_3 \in \mathbb{C}^\times$, each member being isomorphic to $p = p_3 v^3 + \sum_{t \geq 4} p_t v^t$.
- ...

□

3.6.3 Bundles on W_3

On $W_3 := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ the conormal sheaf N_{Z, W_3}^* is not ample, but unlike on W_2 it is not even possible to express every vector bundle as a filtration. In particular, there are rank-2 bundles that are not extensions of line bundles, and the transition functions need not be algebraic (e.g. on the subset $\text{Tot}(0 \oplus \mathcal{O}_{\mathbb{P}^1}(1))$).

Note however that for purely dimensional reasons,

$$h^2(Z; \mathcal{E}nd F \otimes S^n(N_{X, W}^*)) = 0$$

for every vector bundle (in fact, coherent sheaf) F on Z , and that it is thus possible to extend F to a bundle E on $\hat{Z}$ such that $E|_Z = F$; however, just like on W_2 there are now infinitely many non-zero terms in the sum in Equation (1.1), i.e. infinitely many directions in which to extend.

If we only consider rank-2 bundles that are extensions of the form (2.1), we still know from Proposition 3.5 that the space of extensions modulo isomorphisms has a generic set of dimension $4j - 5$.

When $j = 0$ we have $p|_{Z^{(1)}} \equiv 0$ by Proposition 3.1. However, there are many non-equivalent bundles on W_2 even for $j = 0$. For $j = 1$, we have a one-dimensional family of generic extensions given by $p_{100}v + p_{1,0,-1}z^{-1}v$ (modulo projectivisation).

Example. The bundle E on W_3 given by the transition matrix

$$T = \begin{pmatrix} 1 + v & z^{-1}v \\ z^{-1}v & 1 \end{pmatrix}$$

is not isomorphic to an extension of line bundles, and $E|_Z$ is trivial.

3.6.4 Structure on the moduli

There is *a priori* no inherent structure on our moduli of extensions of line bundles modulo isomorphisms. We define *ad hoc* a topology on $\mathfrak{M}(j)$ by defining

$$\mathfrak{M}(j) := \left(\{ \text{coefficients } p_{trs} \} \subset \mathbb{C}^{\gamma_+} \right) / \text{isomorphisms}$$

and endowing $\mathfrak{M}(j)$ with the quotient topology. This is still not Hausdorff, since for instance the split bundle $\mathcal{O}(-j) \oplus \mathcal{O}(j)$ is “near” every bundle in this topology. Here

$$\gamma_+ = \sum_{n=1}^{\infty} h^1(Z; \mathcal{E}nd F \otimes S^n(N_{Z, W_i}^*)) \leq \infty$$

is the number of coefficients in p according to Proposition 3.1.

In [BGK, Section 4] it was shown that in the analogous situation of bundles on the surfaces $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ there exists numerical invariants, namely the width and height which we defined in (2.5) and (2.6), which decompose $\mathfrak{M}(j)$ into Hausdorff components.

We remarked that in our three-dimensional cases, the analogue of the width always vanishes, and the height is finite only on W_1 .

Generalisations. After having studied numerous examples in detail, we can make a few generalising remarks. If $Z \cong \mathbb{P}^1$ is a line inside any complex space W and $N_{Z, W}^*$ is ample, then as discussed in § 2.2, bundles on an analytic neighbourhood $N(Z)$ are determined on a finite

infinitesimal neighbourhood $Z^{(M)}$, and the situation is modelled on

$$W_{\mathbf{k}} := \text{Tot}(N_{Z,W}) = \text{Tot}\left(\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(-k_i)\right),$$

where $k_i > 0$ for all i . Bundles on $W_{\mathbf{k}}$ are filtered and algebraic by [BGK09, Theorem 3.2]. A rank-2 bundle E on $W_{\mathbf{k}}$ still splits as $\mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$ on Z , and the dimension of the generic set of the moduli space $\mathfrak{M}(j)$ of extensions of splitting type j modulo bundle isomorphisms can be calculated as the γ_1 -term in Equation (1.1). Each surface $D_i := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k_i))$ in $W_{\mathbf{k}}$ is now of the form Z_{k_i} as studied in [BGK], and a bundle E is generic if it is generic on each D_i . Expressed conversely, if a bundle E given by the extension class p , which only has terms of order 1 in the fibre directions, is *not* generic, then its restriction to some D_i will be the split bundle, which can be identified by its numerical invariants according to [BGK]. Among all bundles which have only terms of first order in the fibre directions in their extension class, the generic ones are precisely those for which the sum of all partial invariants $\sum_i (h(E|_{D_i}) + w(E|_{D_i}))$ is minimal.

Theorem 3.27. *There exists a two-parameter family of embeddings $\Phi_{s,t}: \mathfrak{M}(j) \hookrightarrow \mathfrak{M}(j+1)$, $(s,t) \in \mathbb{P}^1 \times \mathbb{P}^1$, such that $\bigcup_{s,t} \Phi_{s,t}(\mathfrak{M}(j)) = \mathfrak{M}(j+1) - S(j+1)$, where $S(j+1)$ is the set of bundles of splitting type $j+1$ that do not split on the second infinitesimal neighbourhood.*

Proof. Suppose E is a bundle on W_1 of splitting type j given by the polynomial $p(z, u, v)$. For $[a_1 : b_1], [a_2, b_2] \in (\mathbb{P}^1)^2$, there is a map

$$\Phi_{[a_1:b_1],[a_2,b_2]}: M(j) \rightarrow M(j+1)$$

which is the composite of two elementary transformations over the divisors $D_i = \{a_i u + b_i v\}$, $i = 1, 2$ followed by a twist by $\mathcal{O}(-1)$:

$$\Phi_{[a_1:b_1],[a_2,b_2]}(\mathcal{E}) = (\text{Elm}_{D_2} \circ \text{Elm}_{D_1})(\mathcal{E}) \otimes \mathcal{O}(-1)$$

If the bundle $\mathcal{E}$ is of splitting type j and given by the polynomial p , then $\mathcal{E}' := \Phi_{[a_1:b_1],[a_2,b_2]}(\mathcal{E})$ is given by $z(a_1 u + b_1 v)(a_2 u + b_2 v)p$. Furthermore, $\mathcal{E}'|_Z \cong \mathcal{O}_{\mathbb{P}^1}(-j-1) \oplus \mathcal{O}_{\mathbb{P}^1}(j+1)$, so $\mathcal{E}'$ is of splitting type $j+1$, and $\mathcal{E}'$ fits into the exact sequence

$$0 \longrightarrow \mathcal{O}(-j-1) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{O}(j+1) \longrightarrow 0.$$

By construction, $\mathcal{E}'|_{Z^{(2)}} \cong \mathcal{O}_{Z^{(2)}}(-j-1) \oplus \mathcal{O}_{Z^{(2)}}(j+1)$, that is, bundles in the image split on the second infinitesimal neighbourhood of Z .

It can be seen by direct computation that every bundle of splitting type $j+1$ which splits on $Z^{(2)}$ is in the image of $\Phi_{[a_1:b_1],[a_2,b_2]}$. For this, one observes that every polynomial in the canonical form from Proposition 3.1 which is of total (u, v) -degree ≥ 3 and of splitting type $j+1$ can be written as $z(a_1 u + b_1 v)(a_2 u + b_2 v)q$ for some other polynomial q of splitting type j . $\square$

Bibliography

- [BGK] Eduardo Ballico, Elizabeth Gasparim, and Thomas Köppe, *Vector bundles near negative curves: moduli and local Euler characteristic*, to appear in Comm. Algebra.
- [BGK09] ———, *Local moduli of holomorphic bundles*, J. Pure Appl. Algebra **213** (2009), 397–408.
- [Bla96] Raimund Blache, *Chern classes and Hirzebruch-Riemann-Roch theorem for coherent sheaves on complex-projective orbifolds with isolated singularities*, Math. Z. **222** (1996), no. 1, 7–57.
- [dJP00] Theo de Jong and Gerhard Pfister, *Local analytic geometry: Basic theory and applications*, Vieweg Verlag, 2000.
- [FK74] O. Forster and K. Knorr, *Über die Deformationen von Vektorraumbündeln auf kompakten komplexen Räumen*, Math. Ann. **209** (1974), 291–346.
- [Gas97] Elizabeth Gasparim, *Holomorphic bundles on $\mathcal{O}(-k)$ are algebraic*, Comm. Algebra **25** (1997), no. 9, 3001–3009.
- [GKM08] Elizabeth Gasparim, Thomas Köppe, and Pushan Majumdar, *Local holomorphic Euler characteristic and instanton decay*, Pure Appl. Math. Q. **4** (2008), no. 2, 161–179, Special Issue: In honor of Fedya Bogomolov, Part 1.
- [Gro61] Alexander Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I.*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 5–167.
- [Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52, Springer, 1977.
- [Jim92] Jesús Jiménez, *Contraction of nonsingular curves*, Duke Math. J. **65** (1992), no. 2, 313–332.
- [Pet81] Thomas Peternell, *Vektorraumbündel in der Nähe von kompakten komplexen Unterräumen*, Math. Ann. **257** (1981), no. 1, 111–134.
- [Pet82] ———, *Vektorraumbündel in der Nähe von exzeptionellen Unterräumen – das Modulproblem*, J. Reine Angew. Math. **336** (1982), 110–123.