ON THE GEOMETRY OF MODULI SPACES OF ANTI-SELF-DUAL CONNECTIONS

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ABSTRACT. Consider a simply connected, smooth, projective, complex surface X. Let $\mathcal{M}_k^f(X)$ be the moduli space of framed irreducible anti-self-dual connections on a principal $SU(2)$-bundle over X with second Chern class $k > 0$, and let $\mathcal{C}_k^f(X)$ be the corresponding space of all framed connections, modulo gauge equivalence. A famous conjecture by M. Atiyah and J. Jones says that the inclusion map $\mathcal{M}_k^f(X) \to \mathcal{C}_k^f(X)$ induces isomorphisms in homology and homotopy through a range that grows with k.

In this paper, we focus on the fundamental group, π_1. When this group is finite or polycyclic-by-finite, we prove that if the π_1-part of the conjecture holds for a surface X, then it also holds for the surface obtained by blowing up X at n points. As a corollary, we get that the π_1-part of the conjecture is true for any surface obtained by blowing up n times the complex projective plane at arbitrary points. Moreover, for such a surface, the fundamental group $\pi_1(\mathcal{M}_k^f(X))$ is either trivial or isomorphic to $\mathbb{Z}_2$.

1. Introduction

1.1. **Motivations.** Let X be a simply connected, compact, oriented, Riemannian 4-manifold, $P_k \to X$ a principal $SU(2)$-bundle over X with second Chern class $k > 0$ (such a bundle is unique up to diffeomorphism), and $\mathcal{M}_k^f(X)$ the moduli space of framed gauge equivalence classes of *irreducible anti-self-dual* connections on P_k. We recall that a connection A on P_k is said to be anti-self-dual (with respect to the Riemannian metric) if it satisfies the Yang–Mills anti-self-duality equation

$$F_A = - * F_A,$$

where F_A is the curvature associated with A and $*$ denotes the Hodge dual. The space $\mathcal{M}_k^f(X)$ includes naturally into the space $\mathcal{C}_k^f(X)$ of *all* framed gauge equivalence classes of connections on P_k. In [1], M. Atiyah and J. Jones conjectured that, for k large enough, the inclusion map $\mathcal{M}_k^f(X) \to \mathcal{C}_k^f(X)$ induces isomorphisms

$$H_q(\mathcal{M}_k^f(X)) \xrightarrow{\sim} H_q(\mathcal{C}_k^f(X)),$$
$$\pi_q(\mathcal{M}_k^f(X), *) \xrightarrow{\sim} \pi_q(\mathcal{C}_k^f(X), *),$$

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in homology and homotopy for $q \leq q(X,k)$, where $q(X,k) \to \infty$ as $k \to \infty$. (The original statement by Atiyah and Jones is for $X = \mathbb{S}^4$ but the question readily generalizes to other 4-manifolds.)

This conjecture was proved by C. Boyer, J. Hurtubise, B. Mann and R. Milgram in [3] when $X = \mathbb{S}^4$, in which case $q(X,k) = [k/2] - 2$. (See also Y. Tian [36] and F. Kirwan [25] for $SU(l)$-connections with $l > 2$.) In [22], J. Hurtubise and R. Milgram proved the homology part of the conjecture when X is a ruled complex surface. In [13], the third author showed that if the homology part of the conjecture holds true for a projective complex surface X, then it also holds true for the surface obtained by blowing up X at n points (in particular, the homology part of the conjecture is true for rational surfaces). Our purpose in the present paper is to give a version of this result for the fundamental group whenever this group is finite or polycyclic-by-finite. The essential technical tool in the proof is a delicate analysis of the effect on homotopy upon removing a closed subvariety (cf. section 4).

1.2. **Statements of the main results.** We are interested in the following π_1-part of the conjecture.

Conjecture 1.1 (Atiyah–Jones)**.** *Let X be a simply connected, smooth, projective, complex surface. For k large enough, the inclusion map $\mathcal{M}_k^f(X) \to \mathcal{C}_k^f(X)$ induces an isomorphism*

$$\pi_1(\mathcal{M}_k^f(X), *) \xrightarrow{\sim} \pi_1(\mathcal{C}_k^f(X), *).$$

Our main result is as follows.

Theorem 1.2. *Let X be a simply connected, smooth, projective, complex surface, and $\widetilde{X}$ the surface obtained by blowing up X at n points. We suppose that, for any sufficiently large k, the fundamental group $\pi_1(\mathcal{M}_k^f(X))$ is finite or polycyclic-by-finite. Under these conditions, if Conjecture 1.1 is true for the surface X, then it is also true for the surface $\widetilde{X}$.*

We recall that a group G is called *polycyclic-by-finite* if it has a normal polycyclic subgroup of finite index. (One also says that G is an *M-group.*) A group G is *polycyclic* if it has a finite normal series $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_l = G$ such that each factor G_i/G_{i-1} is cyclic. For details and further properties, see [32, 33].

Remark 1.3. When the moduli spaces $\mathcal{M}_k^f(X)$ and $\mathcal{M}_k^f(\widetilde{X})$ are *nilpotent*, Theorem 1.2 extends to higher homotopy groups (even if these groups are not finite or polycyclic-by-finite). Indeed, it is well known that for such spaces the Whitehead theorem holds (cf. [7, §4.3]), and our claim then simply follows from the homology version of [13].

Theorem 1.2 has the following corollary.

Corollary 1.4. *Conjecture 1.1 is true for any surface X obtained by blowing up n times the complex projective plane at arbitrary points. Moreover, in this case, $\pi_1(\mathcal{M}_k^f(X))$ is either trivial or isomorphic to $\mathbb{Z}_2$, provided k is large enough.*

Remark 1.5. Let $\mathcal{S}$ the set of all surfaces X described in Corollary 1.4. They are classically called the surfaces obtained from $\mathbb{P}^2$ by blowing up finitely many points, perhaps infinitely near ones. It obviously contains the set of all surfaces obtained by blowing up the complex projective plane at n points, but it is strictly larger. For instance it contains all surfaces obtained from the Hirzebruch surface F_1 by some blowing up and all surfaces obtained from F_0 (the smooth quadric surface) making at least one blowing up.

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2. From differential to algebraic geometry

As in [3, 22, 13], we shall translate the statements from the introduction, which are in terms of connections, into the language of holomorphic bundles. This is done via the *Kobayashi–Hitchin correspondence*. At the same time, we shall replace the Atiyah–Jones map $\mathcal{M}_k^f(X) \to \mathcal{C}_k^f(X)$ by the so-called *Taubes patching map*.

2.1. **Taubes' patching map and the Kobayashi–Hitchin correspondence.** From now on, X is a simply connected, smooth, projective, complex surface. We denote by $\mathcal{R}_k^f(X)$ the subspace of $\mathcal{C}_k^f(X)$ corresponding to anti-self-dual (both irreducible and reducible) connections. (In particular $\mathcal{R}_k^f(X) \supseteq \mathcal{M}_k^f(X)$.) In [34, 35], C. Taubes constructed a map (so-called Taubes patching map)

$$T_k\colon (\mathcal{C}_k^f(X), \mathcal{R}_k^f(X)) \to (\mathcal{C}_{k+1}^f(X), \mathcal{R}_{k+1}^f(X))$$

satisfying the following two properties (cf. [35, Theorems 2 and 2*]):

(1) $T_k\colon \mathcal{C}_k^f(X) \to \mathcal{C}_{k+1}^f(X)$ is a homotopy equivalence;
(2) (Stable version of the Atiyah–Jones conjecture). If $\mathcal{C}_\infty^f(X)$ and $\mathcal{R}_\infty^f(X)$ are the direct limits of the direct systems $\{\mathcal{C}_k^f(X), T_k\}_{k>0}$ and $\{\mathcal{R}_k^f(X), T_k\}_{k>0}$ respectively, then the inclusion map
$$\mathcal{R}_\infty^f(X) \to \mathcal{C}_\infty^f(X)$$
is a weak homotopy equivalence.

Remark 2.1. Notice that, in general, $T_k(\mathcal{M}_k^f(X))$ is not contained in $\mathcal{M}_{k+1}^f(X)$. It is only a subset of $\mathcal{R}_{k+1}^f(X)$.

Let $\mathfrak{B}_k^f(X)$ be the moduli space of framed rank-2 holomorphic vector bundles over X with Chern classes $c_1 = 0$ and $c_2 = k$. In this moduli space, we consider the subsets $\mathfrak{M}_k^f(X)$ and $\mathfrak{P}_k^f(X)$ consisting of *H-stable* and *H-polystable* bundles respectively, where H is any polarization. By the Kobayashi–Hitchin correspondence (cf. [6, 28, 29] and references therein), we know that there is a homeomorphism of pairs

$$\Phi_k \colon (\mathcal{R}_k^f(X), \mathcal{M}_k^f(X)) \xrightarrow{\sim} (\mathfrak{P}_k^f(X), \mathfrak{M}_k^f(X)).$$

Lemma 2.2. *The inclusion map $i_k \colon \mathfrak{M}_k^f(X) \to \mathfrak{P}_k^f(X)$ induces an isomorphism*

$$(i_k)_\sharp \colon \pi_1(\mathfrak{M}_k^f(X), *) \xrightarrow{\sim} \pi_1(\mathfrak{P}_k^f(X), *),$$

provided k is large enough.

Lemma 2.2 is a consequence of Proposition 5.1 below. The proof of this proposition is not easy. It relies heavily on the rectified homotopy depth [18, 19] of $\mathfrak{P}_k^f(X)$ at strictly polystable points (cf. sections 4 & 5).

By Lemma 2.2, the homomorphism

$$\tau(k) \colon \pi_1(\mathfrak{M}_k^f(X), *) \to \pi_1(\mathfrak{M}_{k+1}^f(X), *),$$

given by

$$\tau(k) := (i_{k+1})_\sharp^{-1} \circ (\Phi_{k+1} \circ T_k \circ \Phi_k^{-1} \circ i_k)_\sharp,$$

is well defined. (As above, the subscript $\sharp$ stands for the homomorphism induced on the fundamental groups.)

Definition 2.3. Hereafter, we shall refer $\tau(k)$ as the *Taubes homomorphism.*

Remark 2.4. Note that, by [15, Main Theorem] or [31, Theorems D & E], the moduli space $\mathfrak{M}_k^f(X)$ is irreducible, and therefore path connected, provided k is large enough. In particular, the base point $*$ can be omitted.

2.2. **The Atiyah–Jones conjecture via holomorphic bundles.** It follows from points (1) and (2) above that the Atiyah–Jones map $\pi_1(\mathcal{M}_k^f(X)) \to \pi_1(\mathcal{C}_k^f(X))$ is an isomorphism for large values of k if and only if $\tau(k)$ is so. We can then restate Conjecture 1.1 in terms of the Taubes homomorphism as follows.

Conjecture 2.5 (Atiyah–Jones)**.** *Let X be a simply connected, smooth, projective, complex surface. For k large enough, the Taubes homomorphism*

$$\tau(k) \colon \pi_1(\mathfrak{M}_k^f(X)) \to \pi_1(\mathfrak{M}_{k+1}^f(X))$$

is an isomorphism.

Theorem 1.2 then takes the following form.

Theorem 2.6. *Let X be a simply connected, smooth, projective, complex surface, and let $\widetilde{X}$ be the surface obtained by blowing up X at n points. We suppose that, for any sufficiently large k, the fundamental group $\pi_1(\mathfrak{M}_k^f(X))$ is finite (respectively, polycyclic-by-finite). Under these conditions, if Conjecture 2.5 is true for the surface X, then the fundamental group $\pi_1(\mathfrak{M}_k^f(\widetilde{X}))$ is finite (respectively, polycyclic-by-finite) and Conjecture 2.5 is true for the surface $\widetilde{X}$ as well.*

Remark 2.7. By induction, it suffices to prove the theorem for $n = 1$. In what follows, we shall then assume that $\widetilde{X}$ is the blow up of X at *one* point.

Similarly, Corollary 1.4 can be restated as follows.

Corollary 2.8. *Conjecture 2.5 is true for any surface X obtained by blowing up n times the complex projective plane at arbitrary points. Moreover, in this case, $\pi_1(\mathfrak{M}_k^f(X))$ is either trivial or isomorphic to $\mathbb{Z}_2$, provided k is large enough.*

The remaining of the paper is devoted to the proofs of Theorem 2.6 and Corollary 2.8.

3. Framings

Recall that we have assumed that $\widetilde{X}$ is the blow-up of X at *one* point (cf. Remark 2.7). Let ℓ be the corresponding exceptional divisor, and $N(\ell)$ be a small neighbourhood of ℓ in $\widetilde{X}$. Since holomorphic bundles on $N(\ell)$ are entirely determined by a finite infinitesimal neighbourhood of the exceptional divisor, and the n-th infinitesimal neighbourhood of ℓ on $\widetilde{X}$ is isomorphic (as a scheme) to the n-th infinitesimal neighbourhood of the exceptional divisor on $\widetilde{\mathbb{C}^2}$, we can assume in what follows that

$$N(\ell) \approx \widetilde{\mathbb{C}^2}.$$

In this paper, we use the concept of framings best suited for a comparison between bundles over X and $\widetilde{X}$ respectively, this means that we wish to define framings such that the following gluing property holds.

Proposition 3.1 (cf. [13, Proposition 4.4])**.** *An isomorphism class $[\widetilde{E^f}]$ of a framed bundle on $\widetilde{X}$ is uniquely determined by a pair of isomorphism classes of framed bundles $[E^f]$ on X and $[V^f]$ on $\widetilde{\mathbb{C}^2}$. We write $\widetilde{E}^f = (E^f, V^f)$.*

For the reader's convenience we summarize here the necessary definitions following [13, Definition 4.3].

Definition 3.2. Let $\pi_F\colon F \to Z$ be a bundle over a surface Z that is trivial over $Z_0 := Z - Y$, where Y is a closed submanifold of Z. Given two pairs $f = (f_1, f_2)\colon Z_0 \to \pi_F^{-1}(Z_0)$ and $g = (g_1, g_2)\colon Z_0 \to \pi_F^{-1}(Z_0)$ of fibrewise linearly independent holomorphic sections of $F|_{Z_0}$, we say that f is *equivalent* to g (written $f \sim g$) if $\phi := g \circ f^{-1}\colon F|_{Z^0} \to F|_{Z^0}$ extends to a holomorphic map $\phi\colon F \to F$ over the entire Z.

(1) A *frame* of F over Z_0 is an equivalence class of fibrewise linearly independent holomorphic sections of F over Z_0. The set of such frames

$$Fram(Z_0, F) := Hol(Z_0, SL(2, \mathbb{C}))/\sim$$

carries the quotient topology.

(2) A *framed* bundle $\widetilde{E}^f$ on $\widetilde{X}$ is a pair consisting of a bundle $\pi_{\widetilde{E}}\colon \widetilde{E} \to \widetilde{X}$ together with a frame of $\widetilde{E}$ over $N^0 := N(\ell) - \ell$.

(3) A *framed* bundle V^f on $\widetilde{\mathbb{C}^2}$ is a pair consisting of a bundle $\pi_V\colon V \to \widetilde{\mathbb{C}^2}$ together with a frame of V over $\widetilde{\mathbb{C}^2} - \ell$.

(4) A *framed* bundle E^f on X is a pair consisting of a bundle $E \to X$ together with a frame of E over $N(x) - \{x\}$, where $N(x)$ is a small disc neighbourhood of x. We will always consider $N(x) = \pi_{\widetilde{E}}(N(\ell))$.

Now let us compare these framings to the case of framings on a divisor. Suppose F is a bundle on the first Hirzebruch surface Σ_1 which is framed over the line at infinity ℓ_∞, that is, we are given F together with a pair of trivializing holomorphic sections σ_{ℓ_∞} of $F|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus 2}$ (here the line $\ell_\infty \subset \Sigma_1$ has positive self-intersection $\ell_\infty^2 = +1$). Then, this pair of sections determines a pair of sections σ_N on a small tubular neighbourhood N of ℓ_∞ in the analytic topology. We know from elementary complex analysis that two holomorphic functions that coincide on a set of points containing an accumulation point are identical, from which it follows that σ_N is uniquely determined. In particular, we can consider the restriction $F|_{\Sigma-\ell_\infty}$ together with the framing $\sigma_N|_{N-\ell_\infty}$. Since $\Sigma_1 - \ell_\infty \simeq \widetilde{\mathbb{C}^2}$ and $N - \ell_\infty \simeq \widetilde{\mathbb{C}^2} - \ell$ we arrive precisely at the concept of framed bundle over $\widetilde{\mathbb{C}^2}$ as defined in (3). For the original surface X, a framing on a pointed disc $N(x) - \{x\}$ is entirely equivalent to a framing on $N(x)$ itself by Hartog's principle. It is also equivalent to a framing on a curve inside $N(x)$ by uniqueness of analytic continuation, so finite dimensionality of the moduli spaces $\mathfrak{M}_k^f(X)$ follows from classical local theory of several complex variables.

We claim that the moduli spaces $\mathfrak{M}_k^f(\widetilde{X})$ of framed bundles over the blown-up surface $\widetilde{X}$ are also finite dimensional. The reasoning goes as follows. Firstly, we consider the deformation theory of bundles on $\widetilde{\mathbb{C}^2}$ (cf. [13, section 6]). In this case, for each given bundle, the space of reframings is a precise quotient obtained from

$$Hol(\widetilde{\mathbb{C}^2} - \ell, sl(2,\mathbb{C}))\Big/ Hol(\widetilde{\mathbb{C}^2}|_{\widetilde{\mathbb{C}^2}-\ell}, sl(2,\mathbb{C}))$$

by analysing meromorphic extensions of sections to the exceptional divisor, and although both numerator and denominator are infinite dimensional, the quotient itself turns out to be finite dimensional. In fact, when we fix the splitting type of the bundle to be j, and take a bundle determined by an extension class $p \in \mathrm{Ext}^1(\mathcal{O}(j), \mathcal{O}(-j))$ then the dimension of the framed local moduli space at p is $m(2j - (m + 1)/2)$, where m is the u-multiplicity of p (cf. [13, Proposition 6.2]). Secondly, as mentioned on the previous paragraph, the initial moduli spaces $\mathfrak{M}_k^f(X)$ are finite dimensional. Thirdly, finite dimensionality of $\mathfrak{M}_k^f(\widetilde{X})$ follows from the stratification that we will present in (7.1). For the moment, let us assume such stratification

$$\mathfrak{M}_k^f(\widetilde{X}) \setminus \mathfrak{S} = \bigcup_{i=0}^{k} \left(\mathfrak{M}_{k-i}^f(X) \times \mathfrak{N}_i^f \right),$$

and focus only a single of its features: it implies that to obtain a bundle $\widetilde{E}$ with second Chern number k on $\widetilde{X}$ we can only use pairs consisting of bundles (E, F), where E is a bundle on X having Chern class $0 \leq c_2(E) = k - i \leq k$ and F a bundle on $\widetilde{\mathbb{C}^2}$ with local holomorphic Euler characteristic i satisfying $0 \leq i = \chi(F) \leq k$; note that here $c_2(\widetilde{E}) = c_2(E) + \chi(F)$. The former moduli spaces are well known, but the latter, the local ones, are rather mysterious. Finite dimensionality of the local moduli follows from the following inequality between the splitting type j and the local holomorphic Euler characteristic $\chi(F)$ of F:

$$j \leq i = \chi(F) \leq j^2.$$

This inequality is shown to be sharp in [14], and in [2] it is shown that all intermediate values do occur, hence all strata of the claimed stratification are non-empty. In particular, we conclude that (7.1) provides a stratification of $\mathfrak{M}_k^f(\widetilde{X})$ all of whose strata are finite dimensional. The claim of finite dimensionality of $\mathfrak{M}_k(\widetilde{X})$ follows.

4. Rectified homotopy depth of complex analytic spaces

As we mentioned before, the rectified homotopy depth will play a crucial role, especially in the proofs of Proposition 5.1 and Lemma 6.1. This section concerns this notion.

Definition 4.1 (Grothendieck [18] and Hamm–Lê [19]). Let A be a complex analytic space, and n an integer.

(1) Let B be a locally closed complex analytic subspace of A. One says that the *rectified homotopy depth of A along B* is greater than or equal to n if for any point $b \in B$ there is a fundamental

system of neighbourhoods $\{U_\alpha\}$ of b in A such that, for any α, the pair $(U_\alpha, U_\alpha \setminus B)$ is $(n-1-\dim_b B)$-connected. That is, for any base point $* \in U_\alpha \setminus B$, the natural map

$$\pi_q(U_\alpha \setminus B, *) \to \pi_q(U_\alpha, *)$$

is bijective for $q \leq n-2-\dim_b B$ and surjective for $q = n-1-\dim_b B$. (Here, $\dim_b B$ is the complex dimension of B at b.)

(2) One says that the *rectified homotopy depth of* A is greater than or equal to n if for any locally closed complex analytic subspace B of A the rectified homotopy depth of A along B is greater than or equal to n.

The following theorem is a homotopy version of a homology vanishing theorem of F. Kirwan [24, Theorem 6.1 and Corollary 6.4]. It is an immediate consequence of the works on homotopy depth due to A. Grothendieck [18], H. Hamm and Lê D. T. [19], and the second author [8]. It will play an important role in the proof of Lemma 6.1.

Theorem 4.2. *Let A be a pure dimensional complex analytic space, B a closed complex analytic subspace of codimension c in A, and m a non-negative integer. We suppose that each point $x_0 \in B$ has a neighbourhood in A which is analytically isomorphic to an open subset of*

$$\{x \in \mathbb{C}^{n_0} \mid f_1(x) = \ldots = f_{m_0}(x) = 0\}$$

for some n_0, m_0 and analytic functions $f_1, \ldots, f_{m_0}$ depending on x_0, with $m_0 \leq m$. Under these conditions, the pair $(A, A \setminus B)$ is $(c-1-m)$-connected. If, moreover, A is locally a complete intersection, then $(A, A \setminus B)$ is $(c-1)$-connected.

Proof. Let x_0 be any point in B and $U(x_0, r)$ a neighbourhood of x_0 in A analytically isomorphic to

$$\{x \in \mathbb{B}(x_0, r) \mid f_1(x) = \ldots = f_{m_0}(x) = 0\}.$$

(Here, $\mathbb{B}(x_0, r)$ is the open ball in $\mathbb{C}^{n_0}$ with centre x_0 and radius r.) By [19, Theorem 3.2.1], the rectified homotopy depth of $U(x_0, r)$ is greater than or equal to $n_0 - m_0$. In particular, the rectified homotopy depth of $U(x_0, r)$ along $B \cap U(x_0, r)$ is greater than or equal to $n_0 - m_0$, and since $B \cap U(x_0, r)$ is a closed subvariety of $U(x_0, r)$, it follows from [8, Théorème 3.11] that

$$(U(x_0, r), U(x_0, r) \setminus B) \tag{4.1}$$

is $(n_0 - m_0 - 1 - \dim B)$-connected. In particular, since $\dim A = \dim_{x_0} A \leq n_0$, this pair is $(c-1-m)$-connected.

We have proved that, for any $r_0 > 0$ sufficiently small, the collection $\{U(x_0, r)\}_{r \leq r_0}$ is a fundamental system of neighbourhoods of x_0 in A such that each pair $(U(x_0, r), U(x_0, r) \setminus B)$, $r \leq r_0$, is $(c-1-m)$-connected. As this is true for each point x_0 in B, it follows from [8, Théorème 2.4] that the pair $(A, A \setminus B)$ is $(c-1-m)$-connected as well. (Note that the pair (A, B) is triangulable by [20, Theorem 3], so [8, Théorème 2.4] applies.)

When A is locally a complete intersection, we have $m_0 = n_0 - \dim A$ and the pair (4.1) is then $(n_0 - (n_0 - \dim A) - 1 - \dim B)$-connected, that is, $(c-1)$-connected. (See also [19, Corollary 3.2.2].) Then, as above, we globalize using [8, Théorème 2.4 or 3.11]. $\square$

5. On the geometry of polystable bundles

The main result of this section is Proposition 5.1. Lemma 2.2 (whose proof has been left temporarily unfinished) is an immediate consequence of this proposition.

Let Y be a simply connected, smooth, projective, complex surface, H an ample divisor on Y, and $\mathfrak{B}^f_k(Y)$ the moduli space of framed rank-2 holomorphic vector bundles over Y with Chern classes $c_1 = 0$ and $c_2 = k$. In this moduli space, we consider the subsets U_1 and U_2 consisting of H-stable and H-polystable bundles respectively, and the subset U_3 corresponding to the H-semistable bundles which are S-equivalent to an H-polystable one. By an H-semistable bundle (respectively, H-polystable or H-stable), we mean a bundle which is H-semistable (respectively, H-polystable or H-stable) as *unframed* bundle.

Proposition 5.1. *For any sufficiently large k, the inclusion maps $\imath_k \colon U_1 \to U_2$ and $\jmath_k \colon U_1 \to U_3$ induce isomorphisms:*

(1) $(\imath_k)_\sharp \colon \pi_1(U_1) \xrightarrow{\sim} \pi_1(U_2)$;
(2) $(\jmath_k)_\sharp \colon \pi_1(U_1) \xrightarrow{\sim} \pi_1(U_3)$.

Remark 5.2. In fact, the proof will show that the homotopy groups $\pi_q(U_1)$ and $\pi_q(U_2)$ (respectively $\pi_q(U_1)$ and $\pi_q(U_3)$) are isomorphic through a range that grows with k.

5.1. **Proof of part (1).** Let $E \in U_2$ be a strictly polystable bundle on Y, that is, E is polystable but not stable. Thus, for some line bundle L we have $E \simeq L \oplus L^*$ with $c_1(E) = 0 = L \cdot H$ and $c_2(E) = k = -L^2$.

Lemma 5.3. *Fix $j \in \mathbb{Z}$, and an effective divisor C in Y. Let $m > 0$ be an integer such that there is a smooth connected curve $D \in |mH|$ and $m(H \cdot H) > \omega_Y(-jC) \cdot H$. Take $E = L \oplus L^*$ with $L \in Pic(Y)$ and $L \cdot H = 0$. Then:*

$$h^2(\mathrm{End}_0(E)(jC)) \leq h^2(jC) + 2 \max\{0, 1 + m(\omega_Y(-jC) \cdot H)\}.$$

Proof. We have

$$h^2(\mathrm{End}_0(E)(jC)) = h^2(jC) + h^2(R(jC)) + h^2(R^*(jC)),$$

where $R := L^{\otimes 2}$ and $R \cdot H = 0$. Then the result follows immediately from the next lemma. $\square$

Lemma 5.4. *Let R be a line bundle on Y with $R \cdot H = 0$.*

(1) *If* $\omega_Y(-jC)\cdot H<0$, *then* $h^2(Y,R(jC))=0$.
(2) *If* $\omega_Y(-jC)\cdot H\geq 0$, *then* $h^2(Y,R(jC))\leq 1+m(\omega_Y(-jC)\cdot H)$.

Proof. By Serre's duality,

$$h^2(Y,R(jC))=h^0(Y,\omega_Y\otimes R^*(-jC)).$$

Consider the exact sequence

$$0\to\omega_Y\otimes R^*(-D)(-jC)\to\omega_Y\otimes R^*(-jC)\to(\omega_Y\otimes R^*(-jC))_{|D}\to 0.$$

Since $H\cdot(\omega_Y\otimes R^*(-D)(-jC))=(-jC\cdot H)(\omega_Y\cdot H)+0+mH\cdot H<0$ and H is ample, we have $h^0(Y,\omega_Y\otimes R^*(-D))=0$. As $(\omega_Y\otimes R^*)_{|D}$ is a line bundle on D of degree $(\omega_Y\otimes R^*)\cdot H=m(\omega_Y\cdot H)$, we have

$$h^0(Y,\omega_Y\otimes R^*(-jC))=0$$

if $\omega_Y(-jC)\cdot H<0$, and

$$h^0(Y,\omega_Y\otimes R^*(-jC))\leq 1+m(\omega_Y(-jC)\cdot H)$$

if $\omega_Y(-jC)\cdot H\geq 0$. □

Lemma 5.5. *Let $V(E)$ be the local deformation space of E, and P the subset of strictly polystable bundles. The pair $(V(E),V(E)\setminus P)$ is n-connected, where*

$$n:=h^1(\mathrm{End}_0(E)(jC))-h^2(\mathrm{End}_0(E)(jC))-1-h^1(\mathcal{O}_Y).$$

Proof. Since E is *simple* as framed bundle, its local deformation space is given by $V(E):=U\cap\psi^{-1}(0)$, where

$$\psi\colon\Big(H^1(\mathrm{End}_0(E)(jC)),0\Big)\to\Big(H^2(\mathrm{End}_0(E)(jC)),0\Big)$$

is the *Kuranishi map* (cf. [6, Proposition 6.4.3]) and U a small neighbourhood of 0 in $H^1(\mathrm{End}_0(E)(jC))$ (cf. [30, §2.1]). Since $H^1(\mathrm{End}_0(E)(jC))$ is smooth, its rectified homotopy depth is simply given by its dimension

$$h^1(\mathrm{End}_0(E)(jC)).$$

It then follows from [19, Theorem 3.2.1] that the rectified homotopy depth of $V(E)$ is greater than or equal to

$$h^1(\mathrm{End}_0(E)(jC))-h^2(\mathrm{End}_0(E)(jC)). \tag{5.1}$$

In particular, the rectified homotopy depth of $V(E)$ along P is greater than or equal to $h^1(\mathrm{End}_0(E)(jC))-h^2(\mathrm{End}_0(E)(jC))$. Then, as P depends at most on $h^1(\mathcal{O}_Y)$ parameters, one deduces from [8, Théorème 3.11] that the pair

$$(V(E),V(E)\setminus P)$$

is n-connected as desired. □

Now we are ready to prove that the inclusion map $\imath_k\colon U_1 \to U_2$ induces isomorphisms in homotopy through a range that grows with k. By Lemma 5.5, for any strictly polystable bundle E, the pair $(V(E), V(E) \setminus P)$ is n-connected. We can then globalize, using [8, Théorème 2.4], and we obtain that the pair (U_2, U_1) is n'-connected, where

$$n' := \inf_{E\in P}\Big(h^1(\mathrm{End}_0(E)(jC)) - h^2(\mathrm{End}_0(E)(jC)) - 1 - h^1(\mathcal{O}_Y)\Big).$$

Now, by [5, §5], we know that, for any $E \in P$, $h^1(\mathrm{End}_0(E)(jC)) \geq \dim V(E) = \dim U_2 = \dim U_1 = 4k - 3(1 + p_g(Y))$, where $p_g(Y)$ is the geometric genus of Y, while Lemma 5.3 says that $h^2(\mathrm{End}_0(E)(jC))$ has an upper bound that does not depend on k. In particular $n' \to +\infty$ as $k \to +\infty$, and the homotopy groups $\pi_q(U_1)$ and $\pi_q(U_2)$ are (naturally) isomorphic through a range that grows with k.

5.2. **Proof of part (2).** Let E be a semistable bundle S-equivalent to a strictly polystable one. Then there is $L \in \mathrm{Pic}(Y)$ such that $E \cong L \oplus L^*$ and $L \cdot H = 0$. It suffices to prove the following lemma (then we conclude as in the previous section).

Lemma 5.6. *Let $V(E)$ be the local deformation space of E, and S the subset of all semistable bundles which are S-equivalent to a strictly polystable one. The pair $(V(E), V(E) \setminus S)$ is n-connected, where n is an integer such that $n \to +\infty$ as $k \to +\infty$.*

Proof. The bundle $L \oplus L^*$ is S-equivalent exactly to the bundles arising in the two families S_1 and S_2, where S_1 is given by all extensions

$$0 \to L^* \to F \to L \to 0,$$

while S_2 is given by all extensions

$$0 \to L \to F \to L^* \to 0.$$

Thus S_1 is given by the split extension plus a projective space of dimension $h^1(R^*) - 1$, while S_2 is the split extension plus a projective space of dimension $h^1(R) - 1$, where $R := L^{\otimes 2}$. As in the proof of Lemma 5.5, we can show that the rectified homotopy depth of $V(E)$ (in particular the rectified homotopy depth of $V(E)$ along S) varies as

$$h^1(R) + h^1(R^*) + d_1 \geq -\chi(R) - \chi(R^*) + d_2,$$

and $h^1(R) \leq -\chi(R) + d_3$, $h^1(R^*) \leq -\chi(R^*) + d_4$, where the d_i's are integers depending only on Y and H (they do not depend on k). Since S depends at most on $h^1(\mathcal{O}_Y)$ parameters, to show that the pair $(V(E), V(E) \setminus S)$ is n-connected with $n \to +\infty$ as $k \to +\infty$, it suffices to prove that the rectified homotopy depth of $V(E)$ along S goes to $+\infty$ as $k \to +\infty$. To do that, it is enough to show that both $-\chi(R)$ and $-\chi(R^*)$ go to $+\infty$ as $k \to +\infty$.

We treat only the case of R, the case of R^* being similar. We have $L \cdot L = -k$, and then $R \cdot R = -4k$. The Riemann–Roch theorem gives

$$\begin{aligned} -\chi(R) &= -(R \cdot R)/2 + (R \cdot \omega_Y)/2 - \chi(\mathcal{O}_Y) \\ &= 2k + (R \cdot \omega_Y)/2 - \chi(\mathcal{O}_Y). \end{aligned}$$

Thus it is sufficient to prove $|\omega_Y \cdot R| \leq \alpha\sqrt{k}$ for some constant α not depending on k. Set $a := (-\omega_Y \cdot H)/(H \cdot H)$ and $b := (\omega_Y + aH) \cdot (\omega_Y + aH)$. These two rational numbers depend only on the pair (Y, H). Notice that $(\omega_Y + aH) \cdot H = 0$. Thus, the ampleness of H and the Hodge index theorem show that the matrix

$$\begin{pmatrix} (\omega_Y + aH) \cdot (\omega_Y + aH) & (\omega_Y + aH) \cdot R \\ (\omega_Y + aH) \cdot R & R \cdot R \end{pmatrix}$$

is semi-definite negative. Thus $b \leq 0$ and $(\omega_Y \cdot R)^2 \leq -4bk$. Since $L \cdot H = 0$, we have $(\omega_Y + aH) \cdot R = \omega_Y \cdot R$. We may take $\alpha := 2\sqrt{b}$. □

Remark 5.7. Notice that when Y is a rational surface, we can always find a polarization H (often called a *good* polarization) for which the moduli space $\mathfrak{B}_k^f(Y)$ is smooth at each point of U_3 and the local deformation space of every H-semistable bundle on Y is smooth.

6. Removing singularities

In general, the moduli spaces $\mathfrak{M}_k^f(X)$ and $\mathfrak{M}_k^f(\widetilde{X})$ that appear in Theorem 2.6 may have singularities. In this section, we show that removing these singularities does not affect the fundamental group. In particular, this will allow us to work only with the smooth parts of $\mathfrak{M}_k^f(X)$ and $\mathfrak{M}_k^f(\widetilde{X})$.

Lemma 6.1. *Let* $\operatorname{Sing}\mathfrak{M}_k^f(X)$ *be the singular locus of* $\mathfrak{M}_k^f(X)$. *For large values of* k, *the inclusion map* $\mathfrak{M}_k^f(X) \setminus \operatorname{Sing}\mathfrak{M}_k^f(X) \to \mathfrak{M}_k^f(X)$ *induces an isomorphism*

$$\pi_1(\mathfrak{M}_k^f(X) \setminus \operatorname{Sing}\mathfrak{M}_k^f(X)) \xrightarrow{\sim} \pi_1(\mathfrak{M}_k^f(X)).$$

(A similar statement also holds for the moduli space $\mathfrak{M}_k^f(\widetilde{X})$.*)*

Remark 6.2. In fact, as above, we will show that removing the singular points does not change the homotopy groups $\pi_q(\mathfrak{M}_k^f(X))$ through a range that grows with k.

Proof of Lemma 6.1. By [27, Theorem 0.1] (see also [16, Theorem 0.3]), we know that, for large k, the moduli space $\mathfrak{M}_k^f(X)$ is locally a complete intersection. By [5, Theorem 5.8], [37, Theorem 2] and [10], we also know that the (complex) codimension c of the singular locus $\operatorname{Sing}\mathfrak{M}_k^f(X)$ in $\mathfrak{M}_k^f(X)$ is greater than or equal to $[k/2]$. Since $\mathfrak{M}_k^f(X)$

is irreducible (cf. [15, Main Theorem] or [31, Theorems D & E]), it is pure dimensional, and it then follows from Theorem 4.2 that the pair

$$(\mathfrak{M}^f_k(X), \mathfrak{M}^f_k(X) \setminus \operatorname{Sing} \mathfrak{M}^f_k(X))$$

is $(c-1)$-connected, and therefore, $([k/2]-1)$-connected. This shows that the homotopy groups $\pi_q(\mathfrak{M}^f_k(X))$ and $\pi_q(\mathfrak{M}^f_k(X) \setminus \operatorname{Sing} \mathfrak{M}^f_k(X))$ are (naturally) isomorphic through a range that grows with k. In particular, for large k, the fundamental groups are the same. □

7. Stratifying the moduli space $\mathfrak{M}^f_k(\widetilde{X})$

By Lemma 6.1, we can work only with the smooth parts of $\mathfrak{M}^f_k(X)$ and $\mathfrak{M}^f_k(\widetilde{X})$ which, by abuse of notation, we still denote by the same symbol.

7.1. **The stratification.** We use the stratification given in [13, Proposition 7.2]. Let us recall briefly the construction. Remember that $\widetilde{X}$ is the blow up of X at *one* point, ℓ the corresponding exceptional divisor, and $N(\ell)$ a neighbourhood of ℓ in the analytic topology so small that we may assume

$$N(\ell) \approx \widetilde{\mathbb{C}^2}$$

(see Remark 2.7 and the beginning of section 3). It follows from Proposition 3.1 that a framed holomorphic bundle $\widetilde{E}$ over $\widetilde{X}$ is uniquely determined by a pair of framed bundles E and F over X and $N(\ell)$ respectively. Here,

$$E = \left(\pi_* \widetilde{E}\right)^{\vee\vee} \quad \text{and} \quad F = \widetilde{E}\mid_{N(\ell)} .$$

In what follows, we shall identify framed bundles over $\widetilde{X}$ with pairs of framed bundles over X and $N(\ell)$ respectively.

Definition 7.1. One says that a bundle F over $N(\ell)$ has *charge* i if the trivial extension to $\widetilde{X}$ (i.e., the pair $\widetilde{E} = (X \times \mathbb{C}^2, F)$) has $c_2(\widetilde{E}) = i$.

We denote by $\mathfrak{N}^f_i$ the space of (isomorphism classes of) framed bundles F over $N(\ell)$ with charge i. By [12, Theorem 5.5], we know that if E is a stable bundle over X with $c_2(E) = k - i$ (i.e., E is in $\mathfrak{M}^f_{k-i}(X)$) and F a bundle over $N(\ell)$ with charge i, then the corresponding bundle $\widetilde{E} = (E, F)$ over $\widetilde{X}$ is also stable (i.e., it belongs to $\mathfrak{M}^f_k(\widetilde{X})$). However, for a given stable bundle $\widetilde{E} = (E, F)$ over $\widetilde{X}$, the bundle E is, in general, only semistable. Therefore, if $\mathfrak{S}$ is the set of bundles $\widetilde{E} = (E, F)$ in $\mathfrak{M}^f_k(\widetilde{X})$ such that E is strictly semistable, then there is a *stratification*

$$\mathfrak{M}^f_k(\widetilde{X}) \setminus \mathfrak{S} = \bigcup_{i=0}^{k} \left(\mathfrak{M}^f_{k-i}(X) \times \mathfrak{N}^f_i \right). \tag{7.1}$$

(Note that if $\widetilde{E} = (E, F)$ is in $\mathfrak{M}^f_k(\widetilde{X}) \setminus \mathfrak{S}$, then $c_2(E) \geq 0$, by Bogomolov's inequality [11, Theorem 9.1], and $c_2(\widetilde{X}) - c_2(X) \geq 0$.)

Remark 7.2. Notice that $\mathfrak{N}^f_0$ is just a point.

7.2. **The homotopy depth of $\mathfrak{M}^f_k(\widetilde{X})$ along $\mathfrak{S}$.** In this section, we show that $\mathfrak{M}^f_k(\widetilde{X})$ and $\mathfrak{M}^f_k(\widetilde{X}) \setminus \mathfrak{S}$ have the same fundamental group. In particular, this will allow us to work only with $\mathfrak{M}^f_k(\widetilde{X}) \setminus \mathfrak{S}$.

Lemma 7.3. *The inclusion map $\mathfrak{M}^f_k(\widetilde{X}) \setminus \mathfrak{S} \to \mathfrak{M}^f_k(\widetilde{X})$ induces an isomorphism*

$$\pi_1(\mathfrak{M}^f_k(\widetilde{X}) \setminus \mathfrak{S}) \xrightarrow{\sim} \pi_1(\mathfrak{M}^f_k(\widetilde{X})),$$

provided k is large enough.

Remark 7.4. Again, the proof will show that the homotopy groups $\pi_q(\mathfrak{M}^f_k(\widetilde{X}) \setminus \mathfrak{S})$ and $\pi_q(\mathfrak{M}^f_k(\widetilde{X}))$ are isomorphic through a range that grows with k.

Proof of Lemma 7.3. For any $0 \leq i \leq k$, the dimension of the set of strictly semistable bundles E over X with $c_2(E) = k - i$ is less than or equal to $(k - i) + 2(k - i) + c$, where c is a constant depending only on X (the term $(k - i)$ comes from (4) in the proof of [13, Proposition 3.2], the term $2(k - i)$ comes from the proof of Lemma 5.6). By [13, Proposition 6.2] the dimension of $\mathfrak{N}^f_i$ is less than or equal to $2i - 1$. Therefore,

$$\dim \mathfrak{S} \leq \sup_{0 \leq i \leq k} (3(k - i) + c + 2i - 1) \leq 3k + c - 1.$$

Now, by [5, §5], we know that

$$\dim \mathfrak{M}^f_k(\widetilde{X}) = 4k - 3(1 + p_g(\widetilde{X})),$$

where $p_g(\widetilde{X})$ is the geometric genus of $\widetilde{X}$. Therefore, the codimension of $\mathfrak{S}$ in $\mathfrak{M}^f_k(\widetilde{X})$ is greater than or equal to $k - 3(1 + p_g(\widetilde{X})) - c + 1$. Since $\mathfrak{M}^f_k(\widetilde{X})$ is smooth, it follows from [9, Théorème 0.3] that the inclusion map $\mathfrak{M}^f_k(\widetilde{X}) \setminus \mathfrak{S} \to \mathfrak{M}^f_k(\widetilde{X})$ induces isomorphisms

$$\pi_q(\mathfrak{M}^f_k(\widetilde{X}) \setminus \mathfrak{S}) \xrightarrow{\sim} \pi_q(\mathfrak{M}^f_k(\widetilde{X}))$$

through a range that grows with k. In particular, for large k, the fundamental groups are the same. □

8. Proof of Theorem 2.6

We recall that, by Lemma 6.1, we can work only with the smooth parts of $\mathfrak{M}^f_k(X)$ and $\mathfrak{M}^f_k(\widetilde{X})$ which, by abuse of notation, we still denote by the same symbol. Moreover, by Lemma 7.3, we can also suppose that $\mathfrak{S} = \emptyset$.

8.1. **The case of finite fundamental groups.** In this section, we assume that, for any sufficiently large k, the fundamental group $\pi_1(\mathfrak{M}^f_k(X))$ is finite. By [13, Lemma 7.4], $\mathfrak{M}^f_k(\widetilde{X}) \setminus (\mathfrak{M}^f_k(X) \times \mathfrak{N}^f_0)$ is a closed analytic subset of $\mathfrak{M}^f_k(\widetilde{X})$ of *real* codimension 2. It then follows from [9, Théorème 0.3] that the inclusion map

$$j_k \colon \mathfrak{M}^f_k(X) \approx \mathfrak{M}^f_k(X) \times \mathfrak{N}^f_0 \to \mathfrak{M}^f_k(\widetilde{X})$$

induces an epimorphism

$$\pi_1(\mathfrak{M}^f_k(X)) \twoheadrightarrow \pi_1(\mathfrak{M}^f_k(\widetilde{X})).$$

In particular, $\pi_1(\mathfrak{M}^f_k(\widetilde{X}))$ is finite. Consider the diagram

$$\begin{tikzcd}
\pi_1(\mathfrak{M}^f_k(X)) \arrow[r, "\tau(k)"] \arrow[d, twoheadrightarrow, "(j_k)_\sharp"'] & \pi_1(\mathfrak{M}^f_{k+1}(X)) \arrow[d, twoheadrightarrow, "(j_{k+1})_\sharp"] \\
\pi_1(\mathfrak{M}^f_k(\widetilde{X})) \arrow[r, "\tilde{\tau}(k)"] & \pi_1(\mathfrak{M}^f_{k+1}(\widetilde{X})),
\end{tikzcd}$$

where $\tau(k)$ and $\tilde{\tau}(k)$ are the Taubes homomorphisms corresponding to X and $\widetilde{X}$ respectively. It follows from [13, Proposition 7.5] that this diagram is commutative. (We recall that we have identified the bundles over $\widetilde{X}$ with the pairs of bundles over X and $N(\ell)$ respectively.) By hypothesis, if k is large enough, then the Taubes homomorphism

$$\tau(k) \colon \pi_1(\mathfrak{M}^f_k(X)) \to \pi_1(\mathfrak{M}^f_{k+1}(X))$$

is an isomorphism. Therefore, for large values of k, the Taubes homomorphism

$$\tilde{\tau}(k) \colon \pi_1(\mathfrak{M}^f_k(\widetilde{X})) \to \pi_1(\mathfrak{M}^f_{k+1}(\widetilde{X}))$$

is an epimorphism, and the sequence

$$k \mapsto |\pi_1(\mathfrak{M}^f_k(\widetilde{X}))|$$

is ultimately decreasing, that is, there exists k_0 such that, for all $k \geq k_0$, we have

$$|\pi_1(\mathfrak{M}^f_k(\widetilde{X}))| \geq |\pi_1(\mathfrak{M}^f_{k+1}(\widetilde{X}))|.$$

(Here, $|\cdot|$ denotes the cardinality.) In particular, this shows that the sequence $k \mapsto |\pi_1(\mathfrak{M}^f_k(\widetilde{X}))|$ is ultimately constant, and the Taubes epimorphism

$$\tilde{\tau}(k) \colon \pi_1(\mathfrak{M}^f_k(\widetilde{X})) \to \pi_1(\mathfrak{M}^f_{k+1}(\widetilde{X}))$$

is then an isomorphism, provided that k is large enough.

8.2. **The case of polycyclic-by-finite fundamental groups.** We now suppose that the group $\pi_1(\mathfrak{M}_k^f(X))$ is polycyclic-by-finite (for any sufficiently large k). As above, we show that the natural map

$$\pi_1(\mathfrak{M}_k^f(X)) \to \pi_1(\mathfrak{M}_k^f(\widetilde{X}))$$

is an epimorphism. This implies, in particular, that the group $\pi_1(\mathfrak{M}_k^f(\widetilde{X}))$ is also polycyclic-by-finite. (Indeed, by [33, 7.1.1] and [32, Exercises 5.4 (11)], we know that every quotient of a polycyclic-by-finite group is also polycyclic-by-finite.) We also show that the hypothesis implies that, for large values of k, there is a sequence of epimorphisms

$$\pi_1(\mathfrak{M}_k^f(\widetilde{X})) \twoheadrightarrow \pi_1(\mathfrak{M}_{k+1}^f(\widetilde{X})) \twoheadrightarrow \pi_1(\mathfrak{M}_{k+2}^f(\widetilde{X})) \twoheadrightarrow \dots$$

given by the Taubes homomorphisms $\tilde{\tau}(k+i)$, $i \geq 0$. Since $\pi_1(\mathfrak{M}_k^f(\widetilde{X}))$ is polycyclic-by-finite, it follows from [26, Lemma 1] that there exists i_0 such that, for $i \geq i_0$,

$$\tilde{\tau}(k+i)\colon \pi_1(\mathfrak{M}_{k+i}^f(\widetilde{X})) \to \pi_1(\mathfrak{M}_{k+i+1}^f(\widetilde{X}))$$

is an isomorphism.

9. Proof of Corollary 2.8

Let $\mathfrak{M}_k'^f(\mathbb{P}^2)$ be the moduli space of rank-2 framed holomorphic H-semistable bundles on $\mathbb{P}^2$ with Chern classes $c_1 = 0$ and $c_2 = k$. By [21, Theorem 3.13 (see also Theorems 3.1 and 3.3)], $\pi_1(\mathfrak{M}_k'^f(\mathbb{P}^2)) \approx \mathbb{Z}_2$.

Lemma 9.1. *The inclusion map* $\mathfrak{M}_k^f(\mathbb{P}^2) \to \mathfrak{M}_k'^f(\mathbb{P}^2)$ *induces an isomorphism*

$$\pi_1(\mathfrak{M}_k^f(\mathbb{P}^2)) \xrightarrow{\sim} \pi_1(\mathfrak{M}_k'^f(\mathbb{P}^2)),$$

provided k is large enough.

Remark 9.2. Once more the proof will give isomorphisms $\pi_q(\mathfrak{M}_k^f(\mathbb{P}^2)) \approx \pi_q(\mathfrak{M}_k'^f(\mathbb{P}^2))$ through a range that grows with k.

Proof of Lemma 9.1. We claim that the codimension of the set of strictly semistable bundles $\mathfrak{S}_k(\mathbb{P}^2)$ in $\mathfrak{M}_k'^f(\mathbb{P}^2)$ is greater than or equal to $k - 3(1 + p_g(\mathbb{P}^2)) - c$, where $p_g(\mathbb{P}^2)$ is the geometric genus of $\mathbb{P}^2$ and c a constant depending only on $\mathbb{P}^2$. Indeed, on one hand, we know that there is a constant c depending only on $\mathbb{P}^2$ such that $\dim \mathfrak{S}_k(\mathbb{P}^2) \leq 3k + c$ (see the proof of Lemma 7.3). On the other hand, by [5, §5], we have

$$\dim \mathfrak{M}_k'^f(\mathbb{P}^2) = \dim \mathfrak{M}_k^f(\mathbb{P}^2) = 4k - 3(1 + p_g(\mathbb{P}^2)).$$

Our claim follows. Now, by [4] and [17] (see also [21, Theorems 3.1 and 3.3]), we know that the moduli space $\mathfrak{M}_k'^f(\mathbb{P}^2)$ is smooth. Applying [9, Théorème 0.3] then gives isomorphisms

$$\pi_q(\mathfrak{M}_k^f(\mathbb{P}^2)) \xrightarrow{\sim} \pi_q(\mathfrak{M}_k'^f(\mathbb{P}^2))$$

through a range that grows with k. $\square$

By [3] or [22] we know that Conjecture 2.5 is true when $X = \mathbb{P}^2$, and by [21] and Lemma 9.1, we have that $\pi_1(\mathfrak{M}^f_k(\mathbb{P}^2))$ is finite (isomorphic to $\mathbb{Z}_2$). The first assertion of Corollary 2.8 then follows from Theorem 2.6. As for the second assertion, we can show as in the proof of Theorem 2.6 that the inclusion map $j_k\colon \mathfrak{M}^f_k(\mathbb{P}^2) \to \mathfrak{M}^f_k(\widetilde{\mathbb{P}^2})$ induces an epimorphism

$$\pi_1(\mathfrak{M}^f_k(\mathbb{P}^2)) \twoheadrightarrow \pi_1(\mathfrak{M}^f_k(\widetilde{\mathbb{P}^2})).$$

By repeating this n times, we get a sequence of natural epimorphisms

$$\pi_1(\mathfrak{M}^f_k(\mathbb{P}^2)) \twoheadrightarrow \pi_1(\mathfrak{M}^f_k(\widetilde{\mathbb{P}^2})) \twoheadrightarrow \pi_1(\mathfrak{M}^f_k(\widetilde{\widetilde{\mathbb{P}^2}})) \twoheadrightarrow \dots \twoheadrightarrow \pi_1(\mathfrak{M}^f_k(X)).$$

As $\pi_1(\mathfrak{M}^f_k(\mathbb{P}^2)) \approx \mathbb{Z}_2$, the fundamental group $\pi_1(\mathfrak{M}^f_k(X))$ is either trivial or isomorphic to $\mathbb{Z}_2$.

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