

HOLOMORPHIC BUNDLES ON $\mathcal{O}(-k)$ ARE ALGEBRAIC

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Abstract

We show that holomorphic bundles on $\mathcal{O}(-k)$ for $k > 0$ are algebraic. We also show that holomorphic bundles on $\mathcal{O}(-1)$ are trivial outside the zero section. A corollary is that bundles on the blow-up of a surface at a point are trivial on a neighborhood of the exceptional divisor minus the exceptional divisor.

1 Introduction

Comparison between analytic and algebraic objects is a classical theme in algebraic geometry and it is always interesting to know about cases when there is no difference between these objects (see Serre's celebrated paper "Géométrie algébrique et géométrie analytique" [6]). We prove that holomorphic bundles

on $\mathcal{O}(-k)$ are algebraic with a view towards applications of these results in the study of bundles over compact surfaces containing these spaces, such as the Hirzebruch surfaces. These applications will appear in a subsequent paper.

In the particular case of $\mathcal{O}(-1)$ we show triviality outside the zero section. Because $\mathcal{O}(-1)$ equals the blow-up of \mathbf{C}^2 at the origin, this result yields an immediate interpretation about bundles over a surface blown-up at a point. Namely, that such bundles are trivial in a neighborhood of the exceptional divisor minus the exceptional divisor. In other words, it means that outside the exceptional divisor every bundle on a blown-up surface \tilde{S}_p is a pull back of a bundle on the surface S .

2 Preliminaires

The line bundle on \mathbf{P}^1 given by the transition function z^k is usually denoted $\mathcal{O}(-k)$. Since we will be studying bundles over this space, we will denote $\mathcal{O}(-k)$ by M_k when we want to view this space as the base of a bundle. We give M_k the charts $M_k = U \cup V$, where $U = \mathbf{C}^2 = \{(z, u)\}$, $V = \mathbf{C}^2 = \{(\xi, v)\}$, $U \cap V = (\mathbf{C} - \{0\}) \times \mathbf{C}$ with change of coordinates $(\xi, v) = (z^{-1}, z^k u)$.

Since $H^1(\mathcal{O}(-k), \mathcal{O}) = 0$, using the exponential sheaf sequence it follows that $Pic(\mathcal{O}(-k)) = \mathbf{Z}$, and holomorphic line bundles on M_k are classified by their Chern classes. Therefore it is clear that holomorphic line bundles over M_k are algebraic. We will denote by $\mathcal{O}^!(j)$ the line bundle on M_k given by transition function z^{-j} .

If E is a rank n bundle over M_k , then over the zero section (which is a \mathbf{P}^1) E splits as a sum of line bundles by Grothendieck's theorem (see [4]). Denoting the zero section by ℓ it follows that for some integers j_i uniquely determined up

to order $E_\ell \simeq \bigoplus_{i=1}^n \mathcal{O}(j_i)$. We will show that such E is an algebraic extension of the line bundles $\mathcal{O}^!(j_i)$.

3 Bundles on $\mathcal{O}(-k)$ are algebraic

This section is a generalization of Theorem 2.1 in [2].

Lemma 3.1 : *Holomorphic bundles on M_k with $k \geq 0$ are extensions of line bundles.*

Proof: We give the proof for rank two for simplicity. The case for rank n is proved by induction on n using similar calculations. Suppose rank $E = 2$ and $E_\ell \simeq \mathcal{O}(-j_1) \oplus \mathcal{O}(-j_2)$. A transition matrix for E from U to V therefore takes the form

$$T = \begin{pmatrix} z^{j_1} + ua & uc \\ ud & z^{j_2} + ub \end{pmatrix}$$

where a, b, c , and d are holomorphic functions in $U \cap V$. We want to change coordinates to obtain an upper triangular transition matrix

$$\begin{pmatrix} z^{j_1} & uc \\ 0 & z^{j_2} \end{pmatrix},$$

which is equivalent to an extension

$$0 \rightarrow \mathcal{O}^!(-j_1) \rightarrow E \rightarrow \mathcal{O}^!(-j_2) \rightarrow 0.$$

If we start with a matrix T where $uc = 0$, then the obvious choice of change of coordinates is to multiply on the right and on the left by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which will make the matrix upper triangular. Now we may assume that $uc \neq 0$ and without loss of generality we may also assume that $j_1 \geq j_2$. Our required change of coordinates will be

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \begin{pmatrix} z^{j_1} + ua & uc \\ ud & z^{j_2} + ub \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}$$

where ξ is a holomorphic function on U and η is a holomorphic function on V whose values will be determined in the following calculations.

After performing this multiplication, the entry e_{21} becomes

$$e_{21} = \eta (z^{j_1} + ua) + ud + [\eta uc + (z^{j_2} + ub)] \xi.$$

We will choose ξ and η to make $e_{21} = 0$. We write the power series expansions for ξ and η as $\xi = \sum_{i=0}^{\infty} \xi_i(z) u^i$ and $\eta = \sum_{i=0}^{\infty} \eta_i(z^{-1}) (z^k u)^i$, and plug into the expression for e_{21} . The term independent of u in e_{21} is

$$\eta_0(z^{-1})z^{j_1} + \xi_0(z)z^{j_2}.$$

Since $j_2 - j_1 \leq 0$ we may choose $\eta_0(z^{-1}) = z^{j_2-j_1}$ and $\xi_0(z) = -1$. After these choices e_{21} is now a multiple of u . Suppose that the coefficients of η and ξ have been chosen up to power u^{n-1} so that e_{21} becomes a multiple of u^n . Then the coefficient of u^n in the expression for e_{21} is

$$\eta_n z^{j_1+kn} + \xi_n z^{j_2} + \Phi$$

where Φ is a holomorphic function of z and z^{-1} . We separate Φ into two parts

$$\Phi = \Phi_{>j_2} + \Phi_{\leq j_2}$$

where $\Phi_{>j_2}$ is the part of Φ containing the powers z^i for $i > j_2$ and $\Phi_{\leq j_2}$ is the part of Φ containing powers z^i for $i \leq j_2$. The appropriate choices of η_n and ξ_n are

$$\eta_n = z^{-j_1-nk} \Phi_{\leq j_2}$$

and

$$\xi_n = -z^{-j_2} \Phi_{>j_2}.$$

These choices cancel the coefficient of u^n in e_{21} . Induction on n gives $e_{21} = 0$.

We get a transition matrix of the form

$$\begin{pmatrix} z^{j_1} + ua & uc \\ 0 & z^{j_2} + ub \end{pmatrix},$$

for suitable a, b, c and d . We must show that the power series defining ξ and η are convergent. Let's see that ξ is a holomorphic function of z and u . We have that $e_{11}e_{22} = \det \begin{pmatrix} e_{11} & e_{12} \\ 0 & e_{22} \end{pmatrix} = \det T$. Therefore $e_{11}e_{22}$ is a holomorphic function in $U \cap V$ which never vanishes. It follows that e_{11} is holomorphic in a dense open subset of $U \cap V$. But $e_{11} = z^{j_1} + ua + \xi uc$, and $z^{j_1} + ua$ is holomorphic in $U \cap V$ since it is an entry of T . Hence ξuc is holomorphic on an open dense subset of $U \cap V$. Now because $\xi = \xi(z, u)$ has only positive powers of z and u , if ξ is divergent on a point $p = (z_0, u_0) \in U \cap V$, then it is also divergent in the entire open subset $A = \{(z, u) \in U \cap V : |z| > z_0, |u| > u_0\}$ (see [3], p.4). Together with the fact that the product ξuc is holomorphic in $U \cap V$ this forces uc to be identically zero in A and consequently in the whole of $U \cap V$ contrary to our assumptions.

To prove that η is holomorphic look at it as a series of positive powers in z^{-1} and $z^k u$ and repeat an analogous reasoning. Now do a similar trick using the change of coordinates

$$\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \begin{pmatrix} z^{j_1} + ua & uc \\ 0 & z^{j_2} + ub \end{pmatrix} \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$$

and choose ξ_1, ξ_2, η_1 and η_2 appropriately to obtain a new transition matrix

$$\begin{pmatrix} z^{j_1} & uc \\ 0 & z^{j_2} \end{pmatrix}.$$

■

Theorem 3.2 : *Holomorphic bundles over M_k , $k > 0$ are algebraic.*

Proof: Let E be a holomorphic bundle over M_k whose restriction to the zero section is $E_\ell \simeq \bigoplus_{i=1}^n \mathcal{O}(-j_i)$, then E has a transition matrix of the form

$$\begin{pmatrix} z^{j_1} & p_{12} & p_{13} & \cdots & \cdots \\ 0 & z^{j_2} & p_{23} & p_{24} & \cdots \\ \vdots & & \vdots & & \\ 0 & \cdots & 0 & z^{j_{n-1}} & p_{n-1,n} \\ 0 & \cdots & & 0 & z^{j_n} \end{pmatrix}$$

from U to V , where p_{ij} are polynomials defined on $U \cap V$.

Once again we will give the detailed proof for the case $n = 2$. The general proof is by induction on n and is essentially the same as for $n = 2$ only notationally uglier. ■

For the case $n = 2$ we restate the theorem giving the specific form of the polynomial.

Theorem 3.3 : *Let E be a holomorphic rank two vector bundle on M_k whose restriction to the zero section is $E_\ell \simeq \mathcal{O}(-j_1) \oplus \mathcal{O}(-j_2)$, with $j_1 \geq j_2$. Then E has a transition matrix of the form*

$$\begin{pmatrix} z^{j_1} & p \\ 0 & z^{j_2} \end{pmatrix}$$

from U to V , where the polynomial p is given by

$$p = \sum_{i=1}^{\lfloor (j_1 - j_2 - 2)/k \rfloor} \sum_{l=ki+j_2+1}^{j_1-1} p_{il} z^l u^i$$

and $p = 0$ if $j_1 < j_2 + 2$.

Proof: Based on the proof of Theorem 3.1 we know that E has a transition matrix of the form

$$\begin{pmatrix} z^{j_1} & uc \\ 0 & z^{j_2} \end{pmatrix}.$$

We are left with obtaining the form of the polynomial p , for which we perform the coordinate changes

$$\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^{j_1} & uc \\ 0 & z^{j_2} \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix},$$

where the coefficients of $\xi = \sum_{i=0}^{\infty} \xi_i(z) u^i$ and $\eta = \sum_{i=0}^{\infty} \eta_i(z^{-1}) (z^k u)^i$, will be chosen appropriately in the following steps. After performing this multiplication, the entry $e(1, 2)$ of the resulting matrix is

$$e(1, 2) = z^{j_1} \xi + uc + z^{j_2} \eta.$$

The term independent of u in the expression for $e(1, 2)$ is $z^{j_1} \xi_0(z) + z^{j_2+k} \eta_0(z^{-1})$. However, we know from the expression for our matrix T (proof of lemma 3.1), that $e(1, 2)$ must be a multiple of u ; accordingly we choose $\xi_0(z) = \eta_0(z^{-1}) = 0$. Placing this information into the above equation, we obtain

$$e(1, 2) = \sum_{n=1}^{\infty} (\xi_n(z) z^{j_1} + c_n(z, z^{-1}) + \eta_n(z^{-1}) z^{j_2+kn}) u^n.$$

Proceeding as we did in the proof of Lemma 2.1, we choose values of ξ_n and η_n to cancel as many coefficients of z and z^{-1} as possible. In this case ξ and η are defined as powers series coming from tails ends of c and consequently are holomorphic because c is. However, here ξ_n appears multiplied by z^{j_1} (and η_n multiplied by z^{j_2+kn}), therefore the optimal choice of coefficients cancels only powers of z^i with $i \geq j_1$ (resp. z^i with $i \leq j_2 + kn$). Consequently, $e(1, 2)$ is left only with terms in z^l for $j_2 + nk < l < j_1$, and we have the expression

$$e(1, 2) = \sum_{i=1}^{\infty} \sum_{l=nk+j_2+1}^{j_1-1} c_{il} z^l u^i.$$

But i may only vary up to the point where $nk + j_2 + 1 \leq j_1 - 1$ and the polynomial p is given by

$$p = \sum_{i=1}^{[(j_1-j_2-2)/k]} \sum_{l=ik+j_2+1}^{j_1-1} p_{il} z^l u^i.$$

■

4 Triviality outside the zero section

From the previous section we know that bundles on M_k are extensions of line bundles. First we have the following lemma.

Theorem 4.1 : *Holomorphic vector bundles on $\mathcal{O}(-1)$ are trivial outside the zero section.*

Proof: From Theorem 2.2 we know that a holomorphic bundle E on $\mathcal{O}(-1)$ is algebraic. Let $E|_{0^c}$ denote the restriction of E to the complement of the zero section and let $\pi : \mathcal{O}(-1) \rightarrow \mathbf{C}^2$ be the blow up map. Then $\pi_*(E|_{0^c})$ is an algebraic bundle over $\mathbf{C}^2 - 0$ and therefore it extends to a coherent sheaf \mathcal{F} over \mathbf{C}^2 . Then the bidual \mathcal{F}^{**} is a reflexive sheaf and as such has singularity set of codimension 3 or more, which implies that \mathcal{F}^{**} is locally free. Moreover, as a bundle on \mathbf{C}^2 it must be trivial. But \mathcal{F}^{**} restricts to $\pi_*(E|_{0^c})$ on $\mathbf{C}^2 - 0$, hence $\pi_*(E|_{0^c})$ is trivial and so is $E|_{0^c}$. ■

Corollary 4.2 *Holomorphic bundles on the blow up of a surface are trivial on a neighborhood of the exceptional divisor minus the and exceptional divisor.*

Proof: Apply Theorem 3.1 to $\tilde{\mathbf{C}}^2 = \mathcal{O}(-1)$. ■

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