

THREE APPLICATIONS OF INSTANTON NUMBERS

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ABSTRACT. We use instanton numbers to: (i) stratify moduli of vector bundles, (ii) calculate relative homology of moduli spaces and (iii) distinguish curve singularities.

1. INTRODUCTION

Instantons on a blow-up have two local numerical invariants, which we name *height* and *width*. Their sum gives the instanton charge. In this paper we present some ways in which this pair of invariants gives finer information than the charge alone. Firstly, we show that instanton numbers give the coarsest stratification of moduli of bundles on blow-ups for which the strata are separated. Secondly, we show that the relative homology $H_2(\mathfrak{M}_k(\tilde{X}), \mathfrak{M}_k(X))$ is nontrivial; where \mathfrak{M}_k denotes moduli of charge k instantons, and \tilde{X} is obtained from X by blowing up a point. This shows that, despite the fact that $\mathfrak{M}_k(X)$ and $\mathfrak{M}_k(\tilde{X})$ have the same dimension, there is a significant topological difference between them. Thirdly, we give examples of analytically distinct curve singularities, which are not distinguished by any of the classical invariants (δ_P , Milnor number, Tjurina number, and multiplicity) but have distinct instanton numbers.

This paper focuses on rank 2 instantons on blown-up surfaces. We are specially interested in the behavior of instantons near an exceptional divisor. We give an explicit construction of instantons on the blow-up of \mathbb{C}^2 at the origin, denoted $\widetilde{\mathbb{C}^2}$. We show that such instantons are determined by the data $\Delta := (j, p, t_\infty)$, formed by an integer j , a polynomial p , and a framing at infinity, that is, a holomorphic map $t_\infty: \mathbb{C}^2 - \{0\} \rightarrow SL(2, \mathbb{C})$. The charge of Δ takes values between j and j^2 depending on p . However, unlike instantons on S^4 , whose charge is given locally by a unique invariant, called the multiplicity, these instantons have two independent local holomorphic invariants. These invariants do not depend on the choice of framing, and can therefore be calculated directly from the algebraic data (j, p) . A Macaulay2 algorithm that calculates the instanton numbers out of this data is available in [8].

The connection between holomorphic vector bundles and instantons is made through the Kobayashi–Hitchin correspondence. In section 2, we use this correspondence to construct instantons on $\widetilde{\mathbb{C}^2}$. In section 3, we use instanton numbers to stratify moduli of bundles on the blown-up plane with a fixed splitting type over the exceptional divisor. In section 4, we consider the moduli spaces $\mathfrak{M}_k(X)$ and $\mathfrak{M}_k(\tilde{X})$ of rank 2 instantons on a compact surface X and on the surface \tilde{X} = the blow up of X at a point, and prove that $H_2(\mathfrak{M}_k(\tilde{X}), \mathfrak{M}_k(X)) \neq 0$. In section 5, we use instanton numbers as invariants of curve singularities. For curves, the trick is as follows. Given a plane curve $p(x, y) = 0$ with singularity at the origin, choose an integer j , and construct an

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instanton with data (j, p) . We then use the numerical invariants of the instanton as analytic invariants of the curve.

2. INSTANTONS ON $\widetilde{\mathbb{C}^2}$

Every rank 2 instanton on $\widetilde{\mathbb{C}^2}$ is determined by a triple $\Delta: = (j, p, t_\infty)$, where j is an integer, p a polynomial and t_∞ a trivialization at infinity. This characterization comes from putting together two results: on one side, the proof due to King [11] of the Kobayashi–Hitchin correspondence over the noncompact surface $\widetilde{\mathbb{C}^2}$ and, on the other side, the description of rank two holomorphic bundles on $\widetilde{\mathbb{C}^2}$ given in [7]. We review these two results.

Instantons on the blown-up plane are naturally identified with instantons on $\overline{\mathbb{C}P^2}$ framed at infinity; this is a simple consequence of the fact that $\overline{\mathbb{C}P^2}$ is the conformal compactification of $\widetilde{\mathbb{C}^2}$. On his Ph.D. thesis, A. King [11] identifies the moduli space $\text{MI}(\widetilde{\mathbb{C}^2}; r, k)$ of instantons on the blown-up plane of rank r and charge k , with the moduli space $\text{MI}(\overline{\mathbb{C}P^2}, \infty; r, k)$ of instantons on $\overline{\mathbb{C}P^2}$, framed at ∞ , whose underlying vector bundle has rank r , and Chern classes $c_1 = 0$ and $c_2 = k$.

On the other hand, consider the Hirzebruch surface Σ_1 , as the canonical complex compactification obtained from $\widetilde{\mathbb{C}^2}$ by adding a line ℓ_∞ at infinity. Essentially *by definition* King identifies the moduli space $\text{MH}(\widetilde{\mathbb{C}^2}; r, k)$ of “stable” holomorphic bundles on $\widetilde{\mathbb{C}^2}$ with rank r and $c_2 = k$ with the moduli space $\text{MH}(\Sigma_1, \ell_\infty; r, k)$ of holomorphic bundles on Σ_1 with a trivialization along ℓ_∞ and whose underlying vector bundle has rank r , $c_1 = 0$ and $c_2 = k$. King then proves the Kobayashi–Hitchin correspondence in this case, namely that the map

$$\text{MI}(\widetilde{\mathbb{C}^2}; r, k) \rightarrow \text{MH}(\widetilde{\mathbb{C}^2}; r, k)$$

given by taking the holomorphic part of an instanton connection is a bijection. Therefore, a rank 2 instanton on $\widetilde{\mathbb{C}^2}$ is completely determined by a rank two holomorphic bundle on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class, together with a trivialization at infinity. The instanton has charge k if and only if the corresponding holomorphic bundle extends to a bundle on Σ_1 trivial on ℓ_∞ having $c_2 = k$. We are led to study holomorphic rank two bundles on $\widetilde{\mathbb{C}^2}$. As shown in [6], holomorphic bundles on $\widetilde{\mathbb{C}^2}$ are algebraic extensions of line bundles; moreover, by [4], if the first Chern class vanishes, then such bundles are trivial on the complement of the exceptional divisor.

Note: Triviality outside the exceptional divisor is very useful and is intrinsically related to the fact that holomorphic bundles on $\widetilde{\mathbb{C}^2}$ are algebraic, cf. [4]. It is of course not true in general that a holomorphic bundle defined only on $\widetilde{\mathbb{C}^2}$ minus the exceptional divisor is trivial; we make essential use of the fact that our bundles/instantons are defined over the entire $\widetilde{\mathbb{C}^2}$.

Now that we have established the equivalence between instantons and bundles, we give an explicit construction of instantons on $\widetilde{\mathbb{C}^2}$. Because of the triviality at infinity, it follows that we have also existence of instantons on any surface containing a \mathbb{P}^1 with self-intersection -1 .

A holomorphic rank 2 bundle E on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class splits over the exceptional divisor as $O(j) \oplus O(-j)$ for some nonnegative integer j , called the

splitting type of the bundle, and, in this case, E is an algebraic extension

$$(2.1) \quad 0 \rightarrow O(-j) \rightarrow E \rightarrow O(j) \rightarrow 0$$

(here by abuse of notation we write $O(k)$ both for the line bundle $O_{\mathbb{P}^1}(k)$ as well as for its pull-back to $\widetilde{\mathbb{C}^2}$). A bundle E fitting in an exact sequence (1) is determined by its extension class $p \in \text{Ext}^1(O(-j), O(j))$. We fix, once and for all, the following coordinate charts:

$$\widetilde{\mathbb{C}^2} = U \cup V$$

where

$$U = \{(z, u)\} \simeq \mathbb{C}^2 \simeq \{(\xi, v)\} = V$$

with

$$(2.2) \quad (\xi, v) = (z^{-1}, zu)$$

in $U \cap V$. Then in these coordinates, the bundle E has a *canonical* transition matrix of the form

$$(2.3) \quad \begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$$

from U to V , where

$$(2.4) \quad p := \sum_{i=1}^{2j-2} \sum_{l=i-j+1}^{j-1} p_{il} z^l u^i$$

is a polynomial in z , z^{-1} and u ([6] Thm. 2.1). Hence E is completely determined by the pair (j, p) . To have an instanton we need also a trivialization at infinity. By [4] Cor. 4.2, E is trivial outside the exceptional divisor. Therefore we may assign to E a trivialization at infinity $t_\infty \in SL(2, \mathbb{C}^2 - \{0\})$ thus obtaining an instanton. As a consequence every rank-two instanton Δ on $\widetilde{\mathbb{C}^2}$ is determined by a triple

$$(2.5) \quad \Delta := (j, p, t_\infty).$$

Generically, two triples (j, p, t_∞) and (j', p', t'_∞) determine the same instanton if and only if $j' = j$, $p' = \lambda p$ and $t'_\infty = A t_\infty$ where $\lambda \neq 0$, and $A \in \Gamma(\mathbb{C}^2 - \{0\}, SL(2, \mathbb{C}))$. To define the topological charge of Δ we need to extend to a compact surface. This (local) charge is independent of the chosen compactification, and in fact only depends on an infinitesimal neighborhood of the exceptional divisor. For simplicity we take the compactification given by the Hirzebruch surface Σ_1 obtained by adding to $\widetilde{\mathbb{C}^2}$ a line at infinity. An instanton Δ on $\widetilde{\mathbb{C}^2}$ corresponds to a bundle E on Σ_1 trivial on ℓ_∞ , together with a trivialization over this line. Let $\pi: \Sigma_1 \rightarrow Z$ be the map that contracts the -1 line. The charge of Δ is by definition

$$(2.6) \quad c(\Delta) = c(E) := c_2(E) - c_2((\pi_* E)^{\vee\vee}).$$

The instanton Δ is generic if and only if its charge equals its splitting type. Moreover, for every $j > 1$ there are nongeneric instantons (j, p, t_∞) , with charge varying from $j + 1$ up to j^2 (see theorem 3.6).

3. LOCAL MODULI SPACES

Two triples $\Delta = (j, p, t_\infty)$ and $\Delta' = (j', p', t'_\infty)$ are equivalent if they represent the same instanton; consequently their corresponding bundles E and E' over $\widetilde{\mathbb{C}^2}$ are isomorphic, hence must have the same splitting type, i.e. $j = j'$. Consider two triples (j, p, t_∞) and (j, p', t'_∞) , with the same j , and corresponding bundles (E, t_∞) and (E', t'_∞) over Σ_1 . An isomorphism of framed bundles is a bundle isomorphism $\Phi: E \rightarrow E'$ such that $\Phi(t_\infty) = t'_\infty$. Two framings t_∞ and t'_∞ for the same underlying bundle E over Σ_1 differ by a holomorphic map $\Phi: \ell_\infty \rightarrow SL(2, \mathbb{C})$ and, since ℓ_∞ is compact, Φ must be constant. Hence, projecting (E, t_∞) on the first coordinate we obtain a fibration of the space of framed bundles over Σ_1 over the space of bundles over Σ_1 which are trivial on the line at infinity, with fibre $SL(2, \mathbb{C})$.

$$(3.1) \quad \begin{array}{c} SL(2, \mathbb{C}) \\ \downarrow \\ \{\text{framed rank } - 2 \text{ bundles over } \Sigma_1\} \\ \downarrow \\ \{\text{rank } - 2 \text{ bundles over } \Sigma_1 \text{ trivial on } \ell_\infty\}. \end{array}$$

We are thus led to study the base space of this fibration. We define M_j to be space of rank two holomorphic bundles on the $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class and with splitting type j , modulo isomorphism, that is,

$$(3.2) \quad M_j = \left\{ \begin{array}{l} E \text{ hol. bundle over } \widetilde{\mathbb{C}^2} : \\ E|_\ell \simeq O(j) \oplus O(-j) \end{array} \right\} / \sim.$$

Fix the splitting type j and set $J = (j-1)(2j-1)$. Then the polynomial p has J coefficients and we identify p with the J -tuple of complex numbers formed by its coefficients written in lexicographical order. We define in \mathbb{C}^J the equivalence relation $p \sim p'$ if (j, p) and (j, p') represent isomorphic bundles. This gives a set-theoretical identification

$$(3.3) \quad M_j = \mathbb{C}^J / \sim.$$

We give \mathbb{C}^J / \sim the quotient topology and M_j the topology induced by (3.3). M_j is generically a complex projective space of dimension $2j-3$ ([6] Thm. 3.5), and is included in M_{j+1} by:

Proposition 3.1. The following map defines a topological embedding

$$\Phi_j : M_j \rightarrow M_{j+1} \\ (j, p) \mapsto (j+1, zu^2p).$$

The proof is in section 5. The map Φ_j takes M_j into the least generic strata of M_{j+1} . In fact, $\text{im}\Phi$ equals the subset of M_{j+1} consisting of all bundles that split on the second formal neighborhood of the exceptional divisor. The complexity of the topology of M_j increases with j . M_2 is non-Hausdorff and by the embedding given in proposition 3.1 this property persists in M_j for $j \geq 2$.

Example 3.2. The description of M_2 as a quotient \mathbb{C}^3 / \sim as in (8) gives $M_2 \simeq \mathbb{P}^1 \cup \{A, B\}$ where points in the generic set \mathbb{P}^1 represent bundles that do not split on the first formal neighborhood. The point A corresponds to a bundle that splits on

the first formal neighborhood but not on higher neighborhoods, and B corresponds to the split bundle (see [6]). The topological counterpart of this decomposition is understood by calculating the instanton charge. If $E \in \mathbb{P}^1$, then $c(E) = 2$ whereas $c(A) = 3$ and $c(B) = 4$.

The good stratification of M_2 by topological invariants agrees with the expectation we might have based on our experience from the case of bundles over compact surfaces. It then appears natural to hope that topological charges stratify M_j into Hausdorff components. This is however entirely false. For each $j > 2$ there are non-Hausdorff subspaces of M_j where the topological charge remains constant. We now define the finer instanton numbers that stratify the spaces M_j into Hausdorff components.

3.1. Instanton numbers. Consider a compact complex surface X together with the blow-up $\pi: \tilde{X} \rightarrow X$ of a point $x \in X$ and denote by ℓ the exceptional divisor. By the Kobayashi–Hitchin correspondence instantons on \tilde{X} (resp. X) correspond to stable bundles on \tilde{X} (resp. X).

Let \tilde{E} be a holomorphic bundle over \tilde{X} satisfying $\det \tilde{E} \simeq O_{\tilde{X}}$ and $\tilde{E}|_{\ell} \simeq O(j) \oplus O(-j)$ with $j \geq 0$. Set $E = (\pi_* \tilde{E})^{\vee\vee}$. Friedman and Morgan ([3], p. 393) gave the following estimate

$$(3.4) \quad j \leq c_2(\tilde{E}) - c_2(E) \leq j^2.$$

Sharpness of these bounds was shown in [5]. Since the n -th infinitesimal neighborhood of ℓ on X is isomorphic (as a scheme) to the n -th infinitesimal neighborhood of ℓ in \mathbb{C}^2 we are able to use the explicit description for bundles on \mathbb{C}^2 , given in (2.3). Hence \tilde{E} is determined on a neighborhood $V(\ell)$ of the exceptional divisor by a pair (j, p) . Define a sheaf Q by the exact sequence,

$$0 \rightarrow \pi_* \tilde{E} \rightarrow E \rightarrow Q \rightarrow 0.$$

Then Q is supported at the point x and $c_2(\pi_* \tilde{E}) - c_2(E) = l(Q)$. An application of Grothendieck–Riemann–Roch gives

$$c_2(\tilde{E}) - c_2(E) = l(Q) + l(R^1 \pi_* \tilde{E}).$$

Both $l(Q)$ and $l(R^1 \pi_* \tilde{E})$ are local analytic invariants and depend only on the data (j, p) defining E over $V(\ell)$. Suppose E is stable on X , then \tilde{E} is stable on \tilde{X} see [3]. Hence, if E corresponds to an instanton on X , then \tilde{E} corresponds to an instanton on \tilde{E} . This justifies the following terminology.

Definition 3.3. A holomorphic bundle \tilde{E} over \tilde{X} such that $\tilde{E}|_{V(\ell)} \simeq \Delta = (j, p, t_\infty)$ and $(\pi_* \tilde{E})^{\vee\vee} \simeq E$ is said to be obtained by obtained by holomorphic patching of Δ to E . If E is given a frame at the point x , then E and Δ uniquely determine \tilde{E} .

Definition 3.4. We set $w(\Delta) := l(Q)$ and $h(\Delta) := h(R^1 \pi_* \tilde{E})$, and call them the height and the width of Δ . The charge of Δ is given by

$$c(\Delta) := \mathbf{w}(\Delta) + \mathbf{h}(\Delta).$$

Remark 3.5. The charge addition given by the patching of Δ can be calculated by a Macaulay2 program [8]. The program has as input j and p and as outputs $\mathbf{w}(\Delta)$ and $\mathbf{h}(\Delta)$.

The following result shows that instanton numbers provide good stratifications for moduli of instantons on $\widetilde{\mathbb{C}}^2$. In fact, these numbers give the coarsest stratification of M_j for which the strata are Hausdorff. In [1] it is shown that the stratification by Chern numbers is not fine enough to have this property.

Theorem 3.6. ([1] Thm. 4.1) *The numerical invariants w and h provide a decomposition $M_j = \cup S_i$ where each S_i is homeomorphic to an open subset of a complex projective space of dimension at most $2j - 3$. For $j > 0$, the lower bounds for these invariants are $(1, j - 1)$ and this pair of invariants takes place on the generic part of M_j which is homeomorphic to $\mathbb{C}P^{2j-3}$ minus a closed subvariety having codimension at least 2. The upper bounds are $(j(j + 1)/2, j(j - 1)/2)$ and this pair occurs at the single point of M_j that represents the split bundle.*

Note that the M_j are labeled by splitting type, however, we also need the loci of fixed local charge i .

Definition 3.7. *The local moduli N_i of bundles with fixed local charge i is*

$$N_i = \left\{ E \text{ hol. bundle over } \widetilde{\mathbb{C}}^2 : c_1(E) = 0, c_2(E) = c_2^{loc}(E) = i \right\} / \sim.$$

Corollary 3.8. *N_0 is just a point, N_1 is also just a point, and $N_2 \simeq \mathbb{C}P^1$. For $i \geq 2$, N_i has dimension $2i - 3$.*

Proof. Just use theorem 3.6 and example 3.2. □

4. TOPOLOGY OF INSTANTON MODULI SPACES

Let X be a compact complex surface. By the Kobayashi–Hitchin correspondence (cf. [13]), we know that irreducible $SU(2)$ instantons of charge k on X are in one-to-one correspondence with rank 2 stable holomorphic bundles on X with Chern classes $c_1 = 0$ and $c_2 = k$. Given a complex surface Y , let $\mathfrak{M}_k(Y)$ denote the moduli of irreducible instantons on Y with charge k , or equivalently, moduli of stable bundles on X having zero first Chern class and second Chern class k .

Let $\pi: \widetilde{X} \rightarrow X$ be the blow up of a point $x \in X$. The aim of this section is to show that there is a significant difference between the moduli spaces of instantons on X and \widetilde{X} , despite the fact that their dimensions coincide. To this purpose we show that $H_2(\mathfrak{M}_k(\widetilde{X}), \mathfrak{M}_k(X)) \neq 0$. To see $\mathfrak{M}_k(X)$ as a subspace of $\mathfrak{M}_k(\widetilde{X})$, the polarizations on the two surfaces have to be chosen appropriately. If L is an ample divisor on X then for large N the divisor $\widetilde{L} = NL - \ell$ is ample on \widetilde{X} . We fix, once and for all, the polarizations L and \widetilde{L} on X and \widetilde{X} respectively. From now on, $\mathfrak{M}_k(Y)$ stands for moduli of rank two bundles on Y slope stable with respect to the fixed polarization. If E is L -stable on X , then $\pi^*(E)$ is \widetilde{L} -stable on \widetilde{X} . Therefore, the pull back map induces an inclusion of moduli spaces $\mathfrak{M}_k(X) \hookrightarrow \mathfrak{M}_k(\widetilde{X})$. We proceed to show that for all $k \geq 1$ the relative homology $H_2(\mathfrak{M}_k(\widetilde{X}), \mathfrak{M}_k(X))$ does not vanish.

We use holomorphic patching as defined in 3.3, and to this end we introduce framings.

Definition 4.1. Framed bundles.

- Let $\pi_F: F \rightarrow Z$ be a bundle over a surface Z that is trivial over $Z_0 := Z - Y$. Given two pairs $f = (f_1, f_2): Z_0 \rightarrow \pi_F^{-1}(Z_0)$ and $g = (g_1, g_2): Z_0 \rightarrow \pi_F^{-1}(Z_0)$ of linearly independent sections of $F|_{Z_0}$, we say that f is equivalent to g if there exist a map holomorphic $\phi: Z_0 \rightarrow SL(2, \mathbb{C})$ satisfying $f = \phi g$ such that

ϕ extends to a holomorphic map over the entire Z . A frame of F over Z_0 is an equivalence class of linearly independent sections over Z_0 .

- A framed bundle \widetilde{E}^f on \widetilde{X} is a pair consisting of a bundle $\pi_{\widetilde{E}}: \widetilde{E} \rightarrow \widetilde{X}$ together with a frame of \widetilde{E} over $N^0 := N(\ell) - \ell$.
- A framed bundle V^f on $\widetilde{\mathbb{C}^2}$ is a pair consisting of a bundle $\pi_V: V \rightarrow \widetilde{\mathbb{C}^2}$ together with a frame of V over $\widetilde{\mathbb{C}^2} - \ell$.
- A framed bundle E on X is a pair consisting of a bundle $E \rightarrow X$ together with a frame of E over $N(x) - x$, where $N(x)$ is a small disc neighborhood of x . We will always consider $N(x) = \pi_{\widetilde{E}}(N(\ell))$.

Notation 4.2. $\mathfrak{M}_k^f(\widetilde{X})$, $\mathfrak{M}_k^f(X)$, and N_i^f denote the framed versions of $\mathfrak{M}_k(\widetilde{X})$, $\mathfrak{M}_k(X)$, and N_i respectively.

For any k , there is a stratification of the moduli space of framed bundles on \widetilde{X} as

$$\mathfrak{M}_k^f(\widetilde{X}) \equiv \bigcup_{i=0}^k \mathfrak{M}_{k-i}^f(X) \times N_i^f.$$

More details of this decomposition are given in [9]. We use the notation

$$K_i := \mathfrak{M}_{k-i}^f(X) \times N_i^f.$$

Lemma 4.3. *Removing the singular points of $\mathfrak{M}_k^f(\widetilde{X})$ does not change homology up to dimension k . That is, if $Sing$ denotes the singularity set of $\mathfrak{M}_k^f(X)$, then for $q < k$*

$$H_q(\mathfrak{M}_k^f(\widetilde{X})) = H_q(\mathfrak{M}_k^f(X) - Sing)$$

Proof. By Kuranishi theory, points $E \in \mathfrak{M}_k^f(\widetilde{X})$ satisfying $H^2(\text{End}_0 E) = 0$ are smooth points. Therefore, the singularity set of $\mathfrak{M}_k^f(\widetilde{X})$ is contained in $\Sigma_k = \{E \in \mathfrak{M}_k^f(\widetilde{X}) : H^2(\text{End}_0 E) = 0\}$. Moreover, the moduli space is defined on a neighborhood of a singular point by $\dim H^2(\text{End}_0 E)$ equations. In ([2] Thm. 5.8), Donaldson shows that $\dim H^2(\text{End}_0 E) \leq a + b\sqrt{k} + 3k$. Therefore, an application of Kirwan's result ([12] Cor. 6.4) gives $H_q(\mathfrak{M}_k^f(\widetilde{X})) = H_q(\mathfrak{M}_k^f(X) - Sing)$ for $q < \dim \mathfrak{M}_k^f(\widetilde{X}) - 2(a + b\sqrt{k} + 3k) < \dim \mathfrak{M}_k^f(X) - 7k = 8k - 3 - 7k < k$. \square

A similar argument holds for $\mathfrak{M}_k^f(X)$. In what follows we work only with the smooth part of $\mathfrak{M}_k^f(\widetilde{X})$ and $\mathfrak{M}_k^f(X)$ which, by abuse of notation, we still denote by the same symbols.

Lemma 4.4. *For $q \leq 2$,*

$$H_q(\mathfrak{M}_k^f(\widetilde{X})) = H_q(K_0 \cup K_1).$$

Proof. The subset of pull-back bundles $K_0 = \{\pi^*(E), E \in \mathfrak{M}_k^f(X)\}$ is well known to be open and dense in $\mathfrak{M}_k^f(\widetilde{X})$. For $i \geq 1$, the subset

$$K_i = \{E \in \mathfrak{M}_k^f(\widetilde{X}) : c_2(\pi_* E^{\vee\vee}) = k - i\} = \mathfrak{M}_{k-i}^f(X) \times N_i^f$$

has real codimension at least $2i$ by [9], Lemma 6.3. We set $S_2 = \bigcup_{i=2}^k K_i$. Then S_2 has codimension at least 4 in $\mathfrak{M}_k^f(\widetilde{X})$. Consequently, using lemma 4.3 for $q < 3$, we have isomorphisms

$$H_q(\mathfrak{M}_k^f(\widetilde{X})) = H_q(\mathfrak{M}_k^f(X) - S_2) = H_q(K_0 \cup K_1).$$

\square

Lemma 4.5. *The real codimension of K_1 in $\mathfrak{M}_k^f(\tilde{X})$ is exactly 2.*

Proof. Since K_0 is open and dense in $\mathfrak{M}_k^f(\tilde{X})$ it follows that the codimension of K_1 in $\mathfrak{M}_k^f(\tilde{X})$ equals the codimension of K_1 inside $K_0 \cup K_1$. By definition, any bundle $\tilde{E} \in K_0$ is trivial around the divisor, and therefore satisfies $\mathbf{w}(\tilde{E}) = \mathbf{h}(\tilde{E}) = 0$. On the other hand, any bundle $\tilde{F} \in K_1$ satisfies $\mathbf{w}(\tilde{F}) = 1$ (by Theorem 3.6). Consequently, $K_1 = \{\tilde{F} \in K_0 \cup K_1 : \mathbf{w}(\tilde{F}) = 1\}$ is the zero locus of a single analytic (in fact algebraic) equation in $K_0 \cup K_1$; hence K_1 has complex codimension one. \square

Theorem 4.6. *Let $k \geq 1$ and suppose $\mathfrak{M}_k^f(X)$ is non-empty, then*

$$H_2(\mathfrak{M}_k^f(\tilde{X}), \mathfrak{M}_k^f(X)) \neq 0.$$

Proof. By lemma 4.4 the map

$$(4.1) \quad H_q(K_0 \cup K_1) \rightarrow H_q(\mathfrak{M}_k^f(\tilde{X}))$$

is an isomorphism, for $q = 0, 1, 2$.

The map of pairs $(K_0 \cup K_1, K_0) \rightarrow (\mathfrak{M}_k^f(\tilde{X}), K_0)$ induces a map between the long exact sequences of these pairs. Using (4.1) and the five lemma we conclude that the map

$$H_2(K_0 \cup K_1, K_0) \rightarrow H_2(\mathfrak{M}_k^f(\tilde{X}), K_0)$$

is an isomorphism. Since K_1 is closed in $K_0 \cup K_1$ we have that

$$H_2(K_0 \cup K_1, K_0) = H_2(\nu(K_1), \nu(K_1) - K_1) = H_2(T\nu(K_1))$$

(by excision), where $\nu(K_1)$ is the normal bundle of K_1 in $K_0 \cup K_1$, and $T\nu(K_1)$ is the Thom space of this bundle. By lemma 4.5, K_1 has codimension exactly 2, therefore the fiber of $T\nu(K_1)$ has dimension 2. Consequently (by the Thom isomorphism or duality Theorems):

$$H_2(T\nu(K_1)) = H_0(K_1) = r\mathbb{Z}$$

where r is the number of components of K_1 , and it follows that $H_2(K_0 \cup K_1, K_0) = r\mathbb{Z}$. If K_1 is connected $H_2(K_0 \cup K_1, K_0) = \mathbb{Z}$. Now, the theorem follows from the simple observation that K_0 is the set of pull-back bundles, which is isomorphic to $\mathfrak{M}_k^f(X)$. \square

Note that in section 2 we constructed instantons on $\widetilde{\mathbb{C}^2}$ with any prescribed charge. However, existence of irreducible instantons on a compact surface follows from existence of the corresponding stable bundles, what bundles in general are only known to exist for large c_2 . On a surface containing a -1 line, however, many nontrivial semistable bundles can be constructed using our holomorphic patching 3.3, by patching any $\widetilde{\mathbb{C}^2}$ instanton bundle to a trivial bundle on X .

5. CURVE SINGULARITIES

Here is how to use instanton numbers to distinguish curve singularities. Start with a curve $p(x, y) = 0$ on \mathbb{C}^2 . Choose your favorite integer j and construct an instanton on $\widetilde{\mathbb{C}^2}$ having data (j, p) . Calculate the height, width and charge of the instanton, use them as invariants of the curve. In other words, we are using the polynomial defining the plane curve as an extension class in $\text{Ext}^1(O(j), O(-j))$. This defines a bundle $E(j, p)$ as in (2.3). We then regard the instanton numbers of this bundle as being associated to the curve.

Note that to perform the computations we must choose a representative for the curve and coordinates for the bundle. Here we use the canonical choice of coordinates for $\widetilde{\mathbb{C}^2}$ as in section 2 and consider only either quasi-homogeneous curves, or else reducible curves which are products of two quasi homogeneous curves. For these curves there is a preferred choice of representative. Whereas this is certainly restrictive, it is nevertheless true that interesting results appear, given that instanton numbers distinguish some of these singularities which are not distinguished by any of the classical invariants: the δ_P invariant, the Milnor number, or the Tjurina number of the singularity (see table II) and in addition the multiplicity (table III).

Taking into account that the blow-up map in our canonical coordinates is given by $x \mapsto u$ and $y \mapsto zu$ the bundle $E(j, p)$ is represented by

$$E(j, p): = \begin{pmatrix} z^j & p(u, zu) \\ 0 & z^{-j} \end{pmatrix}.$$

In this paper we give a few results to illustrate the behavior of the instanton numbers when applied to singularities. Explicit hand-made computations of these invariants for small values of j appear in [1] and [5]. A Macaulay2 algorithm is available to compute the invariants in the general case, see Remark 3.5.

Theorem 5.1. *Instanton numbers distinguish nodes (tacnodes) from cusps (higher order cusps).*

Proof: These singularities have quasi-homogeneous representatives of the form $y^n - x^m$, $n < m$, n even for nodes and tacnodes, and n odd for cusps and higher order cusps. We want to show that instanton numbers detect the parity of the smallest exponent. In fact, more is true, instanton numbers detect the multiplicity itself.

Suppose $n_1 < n_2$. We claim that if $j > n_2$ then $\mathbf{w}(j, p_1) \neq \mathbf{w}(j, p_2)$. In fact, for $n < m$ and large enough j the width takes the value

$$\mathbf{w}(j, y^n - x^m) = n(n+1)/2.$$

Alternatively, by vector bundle reasons we have that $\mathbf{w}(j, p_1) < \mathbf{w}(j, p_2)$. The second assertion is easier to show. The holomorphic bundle $E(j, p_1)$ restricts as a non-trivial extension on the n_1 th formal neighborhood l_{n_1} whereas $E(j, p_2)$ splits on l_{n_1} . These bundle therefore belong to different strata of M_j and by theorem 3.6 must have distinct instanton numbers. \square

We consider the following classical invariants:

- $\delta_P = \dim(\widetilde{O}_P/O_P)$
- Milnor number $\mu = \dim(O/ \langle J(P) \rangle)$
- Tjurina number $\tau = \dim(O/ \langle P, J(P) \rangle)$

Note: The first table is motivated by exercise 3.8 of Hartshorne [10] page 395. However, in the statement of the problem, the first polynomial contains an incorrect exponent. It is written as “ $x^4y - y^4$ ” but it should be “ $x^5y - y^4$.”

TABLE I				$j = 4$	
polynomial	δ_P	μ	τ	\mathbf{w}	h
$x^5y - y^4$	9	17	17	10	6
$x^8 - x^5y^2 - x^3y^2 + y^4$	9	17	15	8	6

Theorem 5.2. *In some cases instanton numbers give finer information than the classical invariants.*

Proof. Table II gives 2 singularities that are obviously distinct, since they have different multiplicities, but are not distinguished by δ_P , Milnor and Tjurina numbers. Table III shows that instanton numbers are the only invariants to distinguish the irreducible singularity $x^3 - x^2y + y^3$ from the reducible singularity $x^3 - x^2y^2 + y^3$. \square

TABLE II					$j = 4$	
polynomial	δ_P	μ	τ	\mathbf{w}	h	
$x^2 - y^7$	3	6	6	3	5	
$x^3 - y^4$	3	6	6	6	6	

TABLE III							$j = 4$	
polynomial	mult.	δ_P	μ	τ	\mathbf{w}	h	charge	
$x^3 - x^2y + y^3$	3	3	4	4	4	3	7	
$x^3 - x^2y^2 + y^3$	3	3	4	4	5	3	8	

Remark 5.3. *The idea of using the polynomial defining a singularity as the extension class of a holomorphic bundle can be further generalized in several ways. For curves themselves, one can use other base spaces. For instance, constructing bundles on the total space of $O_{\mathbb{P}^1}(-k)$ requires very little modifications, but give quite different results. One can also generalize to hypersurfaces in higher dimensions, using the equation of the hypersurface to define an extension of line bundles.*

6. EMBEDDING THEOREM

Proof of Proposition 3.1: We want to show that $(j, p) \mapsto (j + 1, zu^2p)$ defines an embedding $M_j \rightarrow M_{j+1}$. We first show that the map is well defined. Suppose $\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$ and $\begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix}$ represent isomorphic bundles. Then there are coordinate changes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ holomorphic in z, u and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ holomorphic in z^{-1}, zu such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z^{-j} & -p \\ 0 & z^j \end{pmatrix}.$$

Therefore these two bundles are isomorphic exactly when the system of equations

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a + z^{-j}p'c & z^{2j}b + z^j(p'd - ap) - pp'c \\ z^{-2j}c & d - z^{-j}pc \end{pmatrix} \quad (*)$$

can be solved by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ holomorphic in z, u which makes $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ holomorphic in z^{-1}, zu .

On the other hand, the images of these two bundles are given by transition matrices $\begin{pmatrix} z^{j+1} & zu^2p \\ 0 & z^{-j-1} \end{pmatrix}$ and $\begin{pmatrix} z^{j+1} & zu^2p' \\ 0 & z^{-j-1} \end{pmatrix}$, which represent isomorphic bundles iff there are coordinate changes $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ holomorphic in z, u and $\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$ holomorphic in z^{-1}, zu satisfying the equality

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} z^{j+1} & zu^2p' \\ 0 & z^{-j-1} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} z^{-j-1} & -zu^2p \\ 0 & z^{j+1} \end{pmatrix}.$$

That is, the images represent isomorphic bundles if the system

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} \bar{a} + z^{-j}u^2p'\bar{c} & z^{2j+2}\bar{b} + z^{j+2}u^2(p'\bar{d} - \bar{a}p) - z^2u^4pp'\bar{c} \\ z^{-2j-2}\bar{c} & \bar{d} - z^{-j}u^2p\bar{c} \end{pmatrix} \quad (**)$$

has a solution.

Write $x = \sum x_i u^i$ for $x \in \{a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d}\}$ and choose $\bar{a}_i = a_{i+2}$, $\bar{b}_i = b_{i+2}u^2$, $\bar{c}_i = c_{i+2}u^{-2}$, $\bar{d}_i = d_{i+2}$. Then if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ solves (*), one verifies that $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ solves (**), which implies that the images represent isomorphic bundles and therefore Φ_j is well defined. To show that the map is injective just reverse the previous argument. Continuity is obvious. Now we observe also that the image $\Phi_j(M_j)$ is a saturated set in M_{j+1} (meaning that if $y \sim x$ and $x \in \Phi_j(M_j)$ then $y \in \Phi_j(M_j)$). In fact, if $E \in \Phi_j(M_j)$ then E splits in the 2nd formal neighborhood. Now if $E' \sim E$ then E' must also split in the 2nd formal neighborhood therefore the polynomial corresponding to E' is of the form u^2p' and hence $\Phi_j(z^{-1}p')$ gives E' . Note also that $\Phi_j(M_j)$ is a closed subset of M_{j+1} , given by the equations $p_{il} = 0$ for $i = 1, 2$ and $i - j + 1 \leq l \leq j - 1$. Now the fact that Φ_j is a homeomorphism over its image follows from the following easy lemma. \square

Lemma 6.1. *Let $X \subset Y$ be a closed subset and \sim an equivalence relation in Y , such that X is \sim saturated. Then the map $I : X/\sim \rightarrow Y/\sim$ induced by the inclusion is a homeomorphism over the image.*

Proof: Denote by $\pi_X : X \rightarrow X/\sim$ and $\pi_Y : Y \rightarrow Y/\sim$ the projections. Let F be a closed subset of X/\sim . Then $\pi_X^{-1}(F)$ is closed and saturated in X and therefore $\pi_X^{-1}(F)$ is also closed and saturated in Y . It follows that $\pi_Y(\pi_X^{-1}(F))$ is closed in Y/\sim . \square

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