MULTIPLICITY OF COMPLEX HYPERSURFACE SINGULARITIES, ROUCHÉ SATELLITES AND ZARISKI'S PROBLEM

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ABSTRACT. Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be reduced germs of holomorphic functions. We show that f and g have the same multiplicity at 0, if and only if, there exist reduced germs f' and g' analytically equivalent to f and g, respectively, such that f' and g' satisfy a Rouché type inequality with respect to a generic 'small' circle around 0. As an application, we give a reformulation of Zariski's multiplicity question and a partial positive answer to it.

RÉSUMÉ. **Multiplicité des singularités d'hypersurfaces complexes, satellites de Rouché et problème de Zariski.** Soient $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ des germes de fonctions holomorphes réduits. Nous montrons que f et g ont la même multiplicité en 0 si et seulement s'il existe des germes réduits f' et g' analytiquement équivalents à f et g, respectivement, tels que f' et g' satisfassent une inégalité du type de Rouché par rapport à un 'petit' cercle générique autour de 0. Comme application, nous donnons une reformulation de la question de Zariski sur la multiplicité et une réponse partielle positive à celle–ci.

1. Introduction

Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be reduced germs (at the origin) of holomorphic functions, with $n \geq 2$, V_f, V_g the corresponding germs of hypersurfaces in $\mathbb{C}^n$, and ν_f, ν_g the multiplicities at 0 of V_f, V_g respectively. By the *multiplicity* ν_f we mean the number of points of intersection, near 0, of V_f with a generic (complex) line in $\mathbb{C}^n$ passing arbitrarily close to 0 but not through 0. As we are assuming that f is reduced, ν_f is also the *order* of f at 0, that is, the lowest degree in the power series expansion of f at 0. We denote by $C(V_f)$, $C(V_g)$ the tangent cones at 0 of V_f, V_g, that is, the zero sets of the initial polynomials of f and g respectively (cf. [13]).

In Section 2, we prove that $\nu_f = \nu_g$, if and only if, there exist reduced germs f' and g' analytically equivalent to f and g, respectively, such that $|f'(z) - g'(z)| < |f'(z)|$, for all $z \in \dot{D}$, where $\dot{D}$ is the boundary of a generic 'small' disc around 0 (Theorem 2.6). We call such an inequality a *Rouché inequality* and we say that g' is a *Rouché satellite* of f'.

In Section 3, we apply this result to Zariski's multiplicity question. In particular, we show that the answer to Zariski's question is *yes*, if and only if, for any two topologically equivalent reduced germs f and g there exist reduced germs f' and g' analytically equivalent to f and g, respectively, such that g' is a Rouché satellite of f' (Theorem 3.6). In addition, we answer positively Zariski's question in the special case of 'small' homeomorphisms for Newton nondegenerate isolated singularities (Corollary 3.3) and one–parameter families of isolated singularities (Corollary 3.5).

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2. Multiplicity and Rouché satellites

Let L be a line through 0 in $\mathbb{C}^n$ not contained in $C(V_f) \cup C(V_g)$ (equivalently, $L \cap (C(V_f) \cup C(V_g)) = \{0\}$). Then ν_f (respectively ν_g) is the order at 0 of $f_{|L}$ (respectively $g_{|L}$), and 0 is an isolated point of $L \cap V_f$ and $L \cap V_g$ (cf. [2]). In particular, there exists a closed disc $D \subseteq L$ around 0 such that, for any closed disc $D' \subseteq D$ around 0, $D' \cap (V_f \cup V_g) = \{0\}$. We shall call such a disc D a *good* disc for f and for g.

Definition 2.1. We say that g is a *Rouché satellite* of f if there exists a good disc D (for f and for g) such that f and g satisfy a *Rouché inequality* with respect to the boundary $\dot{D}$ of D, that is,

$$|f(z) - g(z)| < |f(z)|$$

for all $z \in \dot{D}$.

Theorem 2.2. *If g is a Rouché satellite of f, then $\nu_g = \nu_f$.*

Proof. Let $D \subseteq L$ be a good disc for f and for g (for some line L through 0 not contained in $C(V_f) \cup C(V_g)$) such that $|f_{|L}(z) - g_{|L}(z)| < |f_{|L}(z)|$ for all $z \in \dot{D}$. By Rouché theorem (cf. e.g. [7, Chapter VI, Theorem 1.6]), $f_{|L}$ and $g_{|L}$ have the same number of zeros, counted with their multiplicities, in the interior of D. Thus, since $f_{|L}$ and $g_{|L}$ vanish only at 0 on D, the orders at 0 of $f_{|L}$ and $g_{|L}$ are equal. In other words, $\nu_f = \nu_g$. $\square$

Example 2.3. Consider the germs f, $g\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ defined by

$$f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^3 + z_1^3 + z_2^4 \quad \text{and} \quad g(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^3 + z_1^4 + z_2^6.$$

Then g is a Rouché satellite of f. Indeed, set $L = \{(z_1, 0, z_3) \in \mathbb{C}^3 \mid z_1 = z_3\}$; then

$$V_f \cap L = \left\{(0,0,0), \left(-\frac{1}{2}, 0, -\frac{1}{2}\right)\right\} \quad \text{and} \quad V_g \cap L = \{(0,0,0), (a, 0, a), (\bar{a}, 0, \bar{a})\},$$

where $a = (-1 - i\sqrt{3})/2$ and $\bar{a}$ is the complex conjugate of a. So, the disc $D \subseteq L$ of radius $1/4$ is good for f and for g, and, for all $z \in \dot{D}$,

$$|f(z) - g(z)| \leq \frac{5}{4^4} < \frac{2}{4^3} \leq |f(z)|.$$

Hence g is a Rouché satellite of f. In fact, here, f is also a Rouché satellite of g. Indeed, for all $z \in \dot{D}$, we have

$$|f(z) - g(z)| \leq \frac{5}{4^4} < \frac{11}{4^4} \leq |g(z)|.$$

Of course, in general, g may be a Rouché satellite of f without f being a Rouché satellite of g. For example, take $g = f/2$. Also, it is not difficult to construct f and g such that $\nu_f = \nu_g$ but neither g is a Rouché satellite of f nor f a Rouché satellite of g. Take for example $g = -f$. Nevertheless, such an unpleasant situation is resolved by Theorem 2.5 below.

Definition 2.4. If there exists a germ of homeomorphism $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that:

(1) $\varphi(V_g) = V_f$ then f and g are called *topologically equivalent* (denoted $f \sim_t g$);
(2) $\varphi(V_g) = V_f$ and φ is an analytic isomorphism, then f and g are called *analytically equivalent* (denoted $f \sim_a g$);
(3) $g = f \circ \varphi$ then f and g are called *topologically right equivalent* (denoted $f \sim_{tr} g$).

Note that the definition makes sense only for *reduced* germs. In the special case of an isolated singularity, the hypothesis '$n \geq 2$' automatically implies that the germ is reduced. Note also that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1).

Theorem 2.2 has the weak following converse.

Theorem 2.5. *If $\nu_f = \nu_g$, then there exist reduced germs $f' \sim_a f$ and $g' \sim_a g$ such that g' is a Rouché satellite of f'.*

Proof. By an analytic change of coordinates, one can assume that the z_n–axis, Oz_n, is not contained in the tangent cones $C(V_f)$, $C(V_g)$, so that $f(0,\ldots,0,z_n) \neq 0$ and $g(0,\ldots,0,z_n) \neq 0$, for any $z_n \neq 0$ close enough to 0. By the Weierstrass preparation theorem, for z near 0, the germ $f(z)$ can be represented as a product $f(z) = f'(z)\, f''(z)$, where $f''(z)$ is a germ of holomorphic function which does not vanish around 0 and where $f'(z)$ is of the form

$$f'(z_1,\ldots,z_n) = z_n^{\nu_f} + z_n^{\nu_f-1} f_1(z_1,\ldots,z_{n-1}) + \ldots + f_{\nu_f}(z_1,\ldots,z_{n-1}),$$

with, for $1 \leq i \leq \nu_f$, $f_i \in \mathbb{C}\{z_1,\ldots,z_{n-1}\}$, $f_i(0) = 0$ and the order of f_i at 0 is $\geq i$. Similarly $g(z) = g'(z)\, g''(z)$, with $g''(z) \neq 0$ for all z near 0, and

$$g'(z_1,\ldots,z_n) = z_n^{\nu_g} + z_n^{\nu_g-1} g_1(z_1,\ldots,z_{n-1}) + \ldots + g_{\nu_g}(z_1,\ldots,z_{n-1}),$$

with, for $1 \leq i \leq \nu_g$, $g_i \in \mathbb{C}\{z_1,\ldots,z_{n-1}\}$, $g_i(0) = 0$ and the order of g_i at 0 is $\geq i$. Clearly f' and g' are reduced, and, since $V_f = V_{f'}$ and $V_g = V_{g'}$, $f' \sim_a f$ and $g' \sim_a g$. On the other hand, since $\nu_f = \nu_g$, $f'_{|Oz_n} = g'_{|Oz_n}$. But for any disc $D \subseteq Oz_n$ around 0 (in particular for any good disc in Oz_n for f' and g'), $|f'(z)| = r^{\nu_f} \neq 0$ for all $z \in \dot{D}$, where r is the radius of D. $\square$

Since the multiplicity is an invariant of the (embedded) reduced analytic type, we can summarize Theorems 2.2 and 2.5 as follows.

Theorem 2.6. *The multiplicities ν_f and ν_g are the same, if and only if, there exist reduced germs $f' \sim_a f$ and $g' \sim_a g$ such that g' is a Rouché satellite of f'.*

3. Applications to Zariski's multiplicity question

In [14], Zariski posed the following question: *if $f \sim_t g$, then is it true that $\nu_f = \nu_g$?* The question is, in general, still unsettled (even for hypersurfaces with isolated singularities). The answer is, nevertheless, known to be *yes* in several special cases the list of which can be found in the recent first author's survey article [3]. In particular, Ephraim [2] proved that multiplicity is preserved by ambient C^1–diffeomorphisms; his paper inspired some of our proofs. In this section, we give a partial positive answer to Zariski's question in the special case of 'small' homeomorphisms for Newton nondegenerate isolated singularities and one–parameter families of isolated singularities. In addition, we give an equivalent reformulation of Zariski's question in terms of Rouché satellites.

We start with the following result which asserts that if f and g are topologically right equivalent via a sufficiently 'small' homeomorphism, then they have the same multiplicity. More precisely suppose $f \sim_{tr} g$. Then there are representatives $\mathrm{f}\colon U \to \mathbb{C}$ and $\mathrm{g}\colon U' \subseteq U \to \mathbb{C}$ of the germs f and g respectively and a homeomorphism $\varphi\colon U' \to \varphi(U') \subseteq U$ such that $\varphi(0) = 0$ and $\mathrm{g} = \mathrm{f} \circ \varphi$. Since f is uniformly continuous on a compact small ball $B_r \subseteq U'$ around 0, there exists $\eta > 0$ such that, for any $z, w \in B_r$,

$$|z - w| < \eta \;\Rightarrow\; |\mathrm{f}(z) - \mathrm{f}(w)| < \inf_{u \in \dot{D}_\varrho} |\mathrm{f}(u)|,$$

where D_ϱ is a good disc at 0 for f and for $\mathrm{g} = \mathrm{f} \circ \varphi$ with radius $\varrho \leq r/2$.

Definition 3.1. We will say that the homeomorphism $\varphi\colon U' \to \varphi(U') \subseteq U$ is f–*small* if there exists a triple (r, ϱ, η) as above such that, for all $z \in B_r$,

$$|z - \varphi(z)| < \inf\{\eta, \varrho\}.$$

Theorem 3.2. *With the above hypotheses and notation, if the homeomorphism* $\varphi\colon U' \to \varphi(U') \subseteq U$ *is* f*–small, then* $\nu_f = \nu_g$.

Proof. By hypothesis, for all $z \in \dot{D}_\varrho$, $\varphi(z) \in B_r$ and

$$|\mathrm{f}(z) - \mathrm{f}\circ\varphi(z)| < \inf_{u\in\dot{D}_\varrho} |\mathrm{f}(u)| \leq |\mathrm{f}(z)|.$$

Therefore $\mathrm{g} = \mathrm{f}\circ\varphi$ is a Rouché satellite of f. Then, by Theorem 2.2, $\nu_f = \nu_g$. □

The interest in topologically right equivalent germs with regard to Zariski's question comes from the following. By theorems of King [4], Perron [8], Saeki [11] and Nishimura [9], if f has an *isolated* singularity at 0 and a nondegenerate Newton principal part, then the relation $f \sim_t g$ implies $f \sim_{tr} g$. On the other hand, by another theorem of King [5], for a one–parameter holomorphic family of *isolated* singularities $(f_s)_s$ in $\mathbb{C}^n$, with $n \neq 3$, if the relation $f_s \sim_t f_0$ holds for all s near 0, then so does $f_s \sim_{tr} f_0$. So, when considering isolated Newton nondegenerate singularities or *families* of isolated singularities, the Zariski problem refers immediately to right equivalent germs.

Corollary 3.3. *Assume that f has an isolated critical point at* 0 *and a nondegenerate Newton principal part, and suppose $g \sim_t f$. In this case, there are representatives* $\mathrm{f}\colon U \to \mathbb{C}$ *and* $\mathrm{g}\colon U' \subseteq U \to \mathbb{C}$ *of f and g respectively and a homeomorphism* $\varphi\colon U' \to \varphi(U') \subseteq U$ *such that* $\varphi(0) = 0$ *and* $\mathrm{g} = \mathrm{f}\circ\varphi$. *If φ is* f*–small, then* $\nu_f = \nu_g$.

Remark 3.4. If, in addition, f is *convenient* (cf. [6]), then the hypothesis of having an isolated singularity at 0 is automatically satisfied (cf. [10]).

Corollary 3.3 is complementary to the result of Abderrahmane and Saia–Tomazella concerning μ–constant *families* of convenient Newton nondegenerate (isolated) singularities (cf. [1] and [12]).

Corollary 3.5. *Let $(f_s)_s$ be a topologically constant (or μ–constant) one–parameter holomorphic family of isolated hypersurface singularities, with $n \neq 3$. In this case, for all s near* 0*, there are representatives* $\mathrm{f}_0\colon U_0 \to \mathbb{C}$ *and* $\mathrm{f}_s\colon U_s \subseteq U_0 \to \mathbb{C}$ *of f_0 and f_s respectively and a homeomorphism* $\varphi_s\colon U_s \to \varphi(U_s) \subseteq U_0$ *such that* $\varphi_s(0) = 0$ *and* $\mathrm{f}_s = \mathrm{f}_0\circ\varphi_s$. *If, for all s near* 0*, φ_s is* f_0*–small, then $(f_s)_s$ is equimultiple (i.e., for all s near* 0*,* $\nu_{f_s} = \nu_{f_0}$*).*

We conclude with the following nice consequence of Theorem 2.6 which is reformulation of Zariski's multiplicity question in terms of Rouché satellites.

Theorem 3.6. *The answer to Zariski's multiplicity question is yes, if and only if, the relation $f \sim_t g$ implies that there exist reduced germs $f' \sim_a f$ and $g' \sim_a g$ such that g' is a Rouché satellite of f'.*

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