SYMPLECTIC LEFSCHETZ FIBRATIONS FROM A LIE THEORETICAL VIEWPOINT

B. CALLANDER, E. GASPARIM, L. GRAMA, L. A. B. SAN MARTIN

ABSTRACT. This is an announcement of results proved in [GGSM1], [GGSM2], [C], and [CG] where methods from Lie theory were used as new tools for the study of symplectic Lefschetz fibrations.

CONTENTS

MOTIVATION

A motivation for studying symplectic Lefschetz fibrations is that, in nice cases, they occur as mirror partners of complex varieties. In fact, given a complex variety Y, the Homological Mirror Symmetry (HMS) conjecture of Kontsevich [Ko] predicts the existence of a symplectic mirror partner X with a superpotential $W\colon X \to \mathbb{C}$. For Fano varieties, the statement of the HMS includes the following: *The category of A-branes* $D(\mathrm{Lag}(W))$ *is equivalent to the derived category of B-branes (coherent sheaves)* $D^b(\mathrm{Coh}(X))$ *on* X. Here $D(\mathrm{Lag}(W))$ is the directed Fukaya–Seidel category of vanishing cycles for the symplectic manifold X and $D^b(\mathrm{Coh}(Y))$ is the bounded derived category of coherent sheaves on Y. An exciting feature of the conjecture is that the A-side is symplectic whereas the B-side is algebraic, and therefore the conjecture provides a dictionary between the two types of geometry – algebraic and symplectic – the mirror map interchanging vanishing cycles on the symplectic side with coherent sheaves on the algebraic side.

HMS has been described in several cases: elliptic curves [PZ], curves of genus two [Se1], curves of higher genus [E], punctured spheres [AAEKO], weighted projective planes and del-Pezzo surfaces [AKO1], [AKO2], quadrics and intersection of two quadrics [S], the four torus [AbS], Calabi–Yau hypersurfaces in projective space [Sh], toric varieties [Ab], Abelian varieties [F], hypersurfaces in toric varieties [AAK], varieties of general type [GKR], and non-Fano toric varieties [BDFKK]. Nevertheless, the HMS conjecture remains open in most cases.

The B-side of the conjecture is better understood in the sense that a lot is known about the category of coherent sheaves on algebraic varieties. In particular, in the

Fano and general type cases, the famous reconstruction theorem of Bondal and Orlov says that you can recover the variety from its derived category of coherent sheaves [BO]. In contrast the A-side is rather mysterious. The intent of this paper is to contribute to the understanding of LG models and subsequently to their categories of vanishing cycles. Using Lie theory, we construct LG models $(\mathscr{O}(H_0), f_H))$, where $\mathscr{O}(H_0)$ is the adjoint orbit of a complex semisimple Lie group and f_H is the height function with respect to an element of the Cartan subalgebra (see Theorem 8).

Even though we had HMS as an encouragement to pursue our work, we do not attempt to prove any instance of it, rather we endeavour to contribute to the understanding of the A-side of the conjecture by describing examples of symplectic Lefschetz fibrations in arbitrary dimensions. We calculate the directed Fukaya–Seidel category in the first nontrivial example, namely the adjoint orbit of $\mathfrak{sl}(2,\mathbb{C})$. For the case of $\mathfrak{sl}(3,\mathbb{C})$ orbits, we discuss (the wild) variations of Hodge diamonds depending on choices of compactifications for our Lefschetz fibrations.

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1. Definitions

DEFINITION 1. A *holomorphic Morse function* on a manifold X is a holomorphic function $f: X \to \mathbb{P}^1$ (or $f: X \to \mathbb{C}$) which has only non-degenerate critical points.

DEFINITION 2. Let X be a complex manifold of dimension n and $f: X \to \mathbb{P}^1$ (or $f: X \to \mathbb{C}$) a surjective holomorphic fibration. We say that f is a *topological Lefschetz fibration* if

(1) there are finitely many critical points $p_1, \ldots, p_k$, and $f(p_i) \neq f(p_j)$ for $i \neq j$;
(2) for each critical point p, there are complex neighbourhoods $p \in U \subset X$, $f(p) \in V \subset \mathbb{P}^1$ on which $f_{|U}$ is represented by the holomorphic Morse function
$$f_{|U}(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2,$$
and such that $\operatorname{crit} f \cap U = \{p\}$; and
(3) the restriction $f_{\text{reg}} := f|_{X - \bigcup X_i}$ to the complement of the singular fibres X_i is a locally trivial fibre bundle.

DEFINITION 3. Let X be a complex manifold and ω a symplectic form making (X, ω) into a symplectic manifold. We say that a topological Lefschetz fibration is a *symplectic Lefschetz fibration* if

(1) the smooth part of any fibre is a symplectic submanifold of (X, ω); and
(2) for each critical point p_i, the form ω_{p_i} is non-degenerate on the tangent cone of X_i at p_i.

2. Non-examples

Proposition 4. *Let M be a compact complex manifold with odd Euler characteristic, then M does not fibre over $\mathbb{P}^1$.*

Proof. The Euler characteristic is multiplicative, that is, for such a fibration we would have $\chi(M) = \chi(\mathbb{P}^1) \cdot \chi(F)$, but $\chi(\mathbb{P}^1) = 2$. □

Corollary 5. [C, cor 2.19] *There are no topological fibrations $f: \mathbb{P}^{2n} \to \mathbb{P}^1$ for $n > 1$.*

Proposition 6. *There are no algebraic fibrations $f: \mathbb{P}^n \to \mathbb{P}^1$ for $n > 1$.*

Proof. Fibres of such a fibration would divisors be in $\mathbb{P}^n$, but by Bezout theorem any two divisors in $\mathbb{P}^n$ intersect. □

3. Good examples in dimension 4

In 4 (real) dimensions, every symplectic manifold admits a Lefschetz fibration after blowing up finitely many points. This is the celebrated result of Donaldson [Do]: *For any symplectic 4-manifold X, there exists a nonnegative integer n such that the n-fold blowup of X, topologically $X\#n\mathbb{CP}^2$, admits a Lefschetz fibration $f\colon X\#n\mathbb{CP}^2 \to S^2$.*

In the opposite direction, still in 4D, the existence of a topological Lefschetz fibration on a symplectic manifold guarantees the existence of a symplectic Lefschetz fibration whenever the fibres have genus at least 2 [GoS]: *If a 4-manifold X admits a genus g Lefschetz fibration $f\colon X \to \mathbb{C}$ with $g \geq 2$, then it has a symplectic structure.*

Moreover, the existence of 4D symplectic Lefschetz fibrations with arbitrary fundamental group is guaranteed by [ABKP]: *Let Γ be a finitely presentable group with a given finite presentation $a\colon \pi_g \to \Gamma$. Then there exists a surjective homomorphism $b\colon \pi_h \to \pi_g$ for some $h \geq g$ and a symplectic Lefschetz fibration $f\colon X \to S^2$ such that the regular fibre of f is of genus h, $\pi_1(X) = \Gamma$, and the natural surjection of the fundamental group of the fibre of f onto the fundamental group of X coincides with $a \circ b$.*

In general it is possible to construct Lefschetz fibrations in 4D starting with a Lefschetz pencil and then blowing up its base locus (see [Se2], [Se3] [Go]). However, in such cases one needs to fix the indefiniteness of the symplectic form over the exceptional locus by glueing in a correction. Direct constructions of Lefschetz fibrations in higher dimensions are by and large lacking in the literature. This gave us our first motivation to investigate the existence of symplectic Lefschetz fibrations on complex n-folds with $n \geq 3$. Our construction does not make use of Lefschetz pencils, we construct our symplectic Lefschetz fibrations directly by taking the height functions that come naturally from the Lie theory viewpoint.

4. A caveat about the norm of complex Morse functions

It is sometimes claimed in the literature that $|f|^2$ is a real Morse function whenever f is a Lefschetz fibration. However, this is in general false. We state this fact as a lemma.

Lemma 7. *Let X be a complex manifold of dimension, $f\colon X \to \mathbb{C}$ a Lefschetz fibration and let p be a critical point of f. Then p is a degenerate critical point of $|f - f(p)|^2$.*

Proof. We may choose (complex) charts centred at p such that with respect to this coordinate system $f(z_1,\dots,z_n) - f(p) = \sum_{i=1}^n z_i^2$. Hence, is it enough to consider the standard Lefschetz fibration $g\colon \mathbb{C}^n \to \mathbb{C}$ given by $g(z_1,\dots,z_n) = \sum_{i=1}^n z_i^2$, and to prove that 0 is a degenerate critical point of $|g|^2$. In real coordinates

$$z := (z_1,\dots,z_n) \mapsto \sum_{i=1}^n z_i^2 = \sum_{i=1}^n x_i^2 - y_i^2 + 2\sqrt{-1}x_i y_i,$$

where we have written $z_i = x_i + \sqrt{-1}y_i$. Then we have the function

$$\begin{aligned} |g|^2 &: \mathbb{C}^n \to \mathbb{R} \\ z &\mapsto \left[\sum_{i=1}^n \left(x_i^2 - y_i^2\right)\right]^2 + 4\left[\sum_{i=1}^n x_i y_i\right]^2 \end{aligned}$$

whose differentials are

$$\partial_{x_k}|g|^2 = 4x_k \sum_{i=1}^n \left(x_i^2 - y_i^2\right) + 8y_k \sum_{i=1}^n x_i y_i,$$

$$\partial_{y_k}|g|^2 = -4y_k \sum_{i=1}^n \left(x_i^2 - y_i^2\right) + 8x_k \sum_{i=1}^n x_i y_i.$$

Since $\operatorname{crit}|g|^2 \supset g^{-1}(0)$, any neighbourhood of 0 contains a non-zero critical point of $|g|^2$ and it follows that 0 is a degenerate critical point of $|g|^2$. □

5. SLFs in higher dimensions via Lie theory

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, and $\mathfrak{h}_{\mathbb{R}}$ the real subspace generated by the roots of $\mathfrak{h}$. An element $H \in \mathfrak{h}$ is called *regular* if $\alpha(H) \neq 0$ for all $\alpha \in \Pi$.

Theorem 8. [GGSM1, thm. 3.1] *Given $H_0 \in \mathfrak{h}$ and $H \in \mathfrak{h}_{\mathbb{R}}$ with H a regular element, the potential $f_H : \mathcal{O}(H_0) \to \mathbb{C}$ defined by*

$$f_H(x) = \langle H, x \rangle \qquad x \in \mathcal{O}(H_0)$$

has a finite number of isolated singularities and defines a Lefschetz fibration; that is to say

1. *the singularities are (Hessian) nondegenerate;*
2. *if $c_1, c_2 \in \mathbb{C}$ are regular values then the level manifolds $f_H^{-1}(c_1)$ and $f_H^{-1}(c_2)$ are diffeomorphic;*
3. *there exists a symplectic form Ω on $\mathcal{O}(H_0)$ such that the regular fibres are symplectic submanifolds;*
4. *each critical fibre can be written as the disjoint union of affine subspaces contained in $\mathcal{O}(H_0)$, each symplectic with respect to Ω.*

The full proof is presented in [GGSM1], a particularly interesting component of the proof states:

Proposition 9. [GGSM1, prop. 3.3] *A point $x \in \mathcal{O}(H_0)$ is a critical point of f_H if and only if $x \in \mathcal{O}(H_0) \cap \mathfrak{h} = \mathcal{W} \cdot H_0$, where $\mathcal{W}$ is the Weyl group.*

Having found a construction of Lefschetz fibrations in higher dimensions, the next step toward a description of the Fukaya–Seidel category of the corresponding LG model would involve the identification of the Fukaya category of a regular fibre. Thus, we studied the diffeomorphism type of a regular level for the Lefschetz fibration. This first required the realisation of the adjoint orbit as the cotangent bundle of a flag manifold, as we now describe.

We choose a set of positive roots Π^+ and simple roots $\Sigma \subset \Pi^+$ with corresponding Weyl chamber is $\mathfrak{a}^+$. A subset $\Theta \subset \Sigma$ defines a parabolic subalgebra $\mathfrak{p}_\Theta$ with parabolic subgroup P_Θ and a flag manifold $\mathbb{F}_\Theta = G/P_\Theta$. An element $H_\Theta \in \operatorname{cl}\mathfrak{a}^+$ is *characteristic* for $\Theta \subset \Sigma$ if $\Theta = \{\alpha \in \Sigma : \alpha(H_\Theta) = 0\}$. Let $Z_\Theta = \{g \in G : \operatorname{Ad}(g)H_\Theta = H_\Theta\}$ be the centraliser in G of the characteristic element H_Θ.

Theorem 10. [GGSM2, thm. 2.1] *The adjoint orbit $\mathcal{O}(H_\Theta) = \operatorname{Ad}(G) \cdot H_\Theta \approx G/Z_\Theta$ of the characteristic element H_Θ is a C^∞ vector bundle over $\mathbb{F}_\Theta$ isomorphic to the cotangent bundle $T^*\mathbb{F}_\Theta$. Moreover, we can write down a diffeomorphism $\iota : \operatorname{Ad}(G) \cdot H_\Theta \to T^*\mathbb{F}_\Theta$ such that*

1. *ι is equivariant with respect to the actions of K, that is, for all $k \in K$,*

   $$\iota \circ \operatorname{Ad}(k) = \widetilde{k} \circ \iota$$

   where K is the compact subgroup in the Iwasawa decomposition $G = KAN$, and $\widetilde{k}$ is the lifting to $T^*\mathbb{F}_\Theta$ (via the differential) of the action of k on $\mathbb{F}_\Theta$.
2. *The pullback of the canonical symplectic form on $T^*\mathbb{F}_\Theta$ by ι is the (real) Kirillov–Kostant–Souriaux form on the orbit.*

Viewing the orbit as the cotangent bundle of a flag manifold, we can identify the topology of the of the fibres in terms of the topology of the flag.

Corollary 11. [GGSM1, cor. 4.5] *The homology of a regular fibre coincides with the homology of* $\mathbb{F}_\Theta \setminus \mathscr{W} \cdot H_\Theta$. *In particular the middle Betti number is* $k-1$ *where* k *is the number of singularities of the fibration (equal to the number of elements in* $\mathscr{W} \cdot H_\Theta$*).*

For the case where singular fibres have only one critical point, we have the following corollary.

Corollary 12. [GGSM1, cor. 5.1] *The homology of the singular fibre though* wH_Θ*,* $w \in \mathscr{W}$*, coincides with that of*

$$\mathbb{F}_{H_\Theta} \setminus \{uH_\Theta \in \mathscr{W} \cdot H_\Theta \mid u \neq w\}.$$

In particular, the middle Betti number of this singular fibre equals $k-2$*, where* k *is the number of singularities of the fibration* f_H*.*

6. Compactifications and their Hodge diamonds

Theorem 10 makes it clear that the adjoint orbits considered here are not compact. We want to compare the behaviour of vanishing cycles on $\mathscr{O}(H_0)$ and on its compactifications. Expressing the adjoint orbit as an algebraic variety, we homogenise its ideal to obtain a projective variety, which serves as a compactification. We calculate the sheaf-cohomological dimensions $\dim H^q(X, \Omega^p)$ for the compactified orbits as well as for the fibres of the SLF. These dimensions shall be called the diamond for the given space; indeed, this is well-known as the Hodge diamond in the non-singular case. Calculating such diamonds is computationally heavy, so we used Macaulay2.

Choosing a compactification is in general a delicate task: a different choice of generators for the defining ideal of the orbit can result in completely different diamonds of the corresponding compactification, as example 1 will show. To illustrate the behaviour of diamonds, we present some examples of adjoint orbits for $\mathfrak{sl}(3,\mathbb{C})$, for which there are three isomorphism types. We chose one that compactifies smoothly and another whose compactification acquires degenerate singularities.

6.1. An SLF with 3 critical values.

In $\mathfrak{sl}(3,\mathbb{C})$, consider the orbit $\mathscr{O}(H_0)$ of

$$H_0 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

under the adjoint action. We fix the element

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

to define the potential f_H. A general element $A \in \mathfrak{sl}(3,\mathbb{C})$ has the form

$$A = \begin{pmatrix} x_1 & y_1 & y_2 \\ z_1 & x_2 & y_3 \\ z_2 & z_3 & -x_1-x_2 \end{pmatrix}. \tag{1}$$

In this example, the adjoint orbit $\mathscr{O}(H_0)$ consists of all the matrices with the minimal polynomial $(A+\mathrm{id})(A-2\,\mathrm{id})$. So the orbit is the affine variety cut out by the ideal I generated by the polynomial entries of $(A+\mathrm{id})(A-2\,\mathrm{id})$. To obtain a projectivisation of X, we first homogenise its ideal I with respect to a new variable t, then take the corresponding projective variety. In this case, the projective variety $\overline{X}$ is a smooth compactification of X and has Hodge diamond:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&0&&0&&& \\ &&0&&2&&0&& \\ &0&&0&&0&&0& \\ 0&&0&&3&&0&&0 \\ &0&&0&&0&&0& \\ &&0&&2&&0&& \\ &&&0&&0&&& \\ &&&&1&&&& \end{array}.$$

We now calculate the Hodge diamond of a compactified regular fibre. The potential corresponding to our choice of H is $f_H = x_1 - x_2$. The critical values of this potential are ± 3 and 0. Since all regular fibres of an SLF are isomorphic, it suffices to chose the regular value 1. We then define the regular fibre X_1 as the variety in $\mathfrak{sl}(3,\mathbb{C}) \cong \mathbb{C}^8$ corresponding to the ideal J obtained by summing I with the ideal generated by $f_H - 1$. We then homogenise J to obtain a projectivisation $\overline{X}_1$ of the regular fibre X_1. The Hodge diamond of $\overline{X}_1$ is:

$$\begin{array}{ccccccc} &&&1&&& \\ &&0&&0&& \\ &0&&2&&0& \\ 0&&0&&0&&0 \\ &0&&2&&0& \\ &&0&&0&& \\ &&&1&&& \end{array}.$$

Remark 1. An interesting feature to observe here is the absence of middle cohomology for the regular fibre. Suppose that the potential extended to this compactification without degenerate singularities, then because f_H has singularities, the fundamental lemma of Picard–Lefschetz theory would imply that there must exist vanishing cycles, which contradicts the absence of middle homology.

Generalising this example to the case of $\mathfrak{sl}(n,\mathbb{C})$, we obtained:

Proposition 13. [CG, Prop. 2] *Let* $H_0 = \mathrm{Diag}(n,-1,\dots,-1)$. *Then the orbit of* H_0 *in* $\mathfrak{sl}(n+1,\mathbb{C})$ *compactifies holomorphically to a trivial product.*

Corollary 14. [CG, Cor 3] *Choose* $H = \mathrm{Diag}(1,-1,0,\dots,0)$ *and* $H_0 = \mathrm{Diag}(n,-1,\dots,-1)$ *in* $\mathfrak{sl}(n+1,\mathbb{C})$. *Any extension of the potential* f_H *to the compactification* $\mathbb{P}^n \times \mathbb{P}^{n*}$ *of the orbit* $\mathcal{O}(H_0)$ *cannot be of Morse type; that is, it must have degenerate singularities.*

6.2. **An SLF with 4 critical values.** In $\mathfrak{sl}(3,\mathbb{C})$ we take

$$H = H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is regular since it has 3 distinct eigenvalues. Then $X = \mathcal{O}(H_0)$ is the set of matrices in $\mathfrak{sl}(3,\mathbb{C})$ with eigenvalues $1, 0, -1$. This set forms a submanifold of real dimension 6 (a complex threefold). In this case $\mathcal{W} \simeq S_3$ acts via conjugation by permutation matrices. Therefore, the potential $f_H = x_1 - x_2$ has 6 singularities; namely, the 6 diagonal matrices with diagonal entries $1, 0, -1$. The four singular values of f_H are $\pm 1, \pm 2$. Thus, 0 is a regular value for f_H. Let $A \in \mathfrak{sl}(3,\mathbb{C})$ be a general element written as in (1), and let $p = \det(A)$, $q = \det(A - \mathrm{id})$. The ideals $\langle p, q\rangle$ and $\langle p-q, q\rangle$ are clearly identical and either of them defines the orbit though H_0 as an affine variety in $\mathfrak{sl}(3,\mathbb{C})$. Now

$$I = \langle p, q, f_H\rangle \qquad J = \langle p, p-q, f_H\rangle$$

are two identical ideals cutting out the regular fibre X_0 over 0. Let I_{hom} and J_{hom} be the respective homogenisations and notice that $I_{\text{hom}} \neq J_{\text{hom}}$, so that they define distinct projective varieties, and thus two distinct compactifications

$$\overline{X}_0^I = \operatorname{Proj}(\mathbb{C}[x_1,x_2,y_1,y_2,y_3,z_1,z_2,z_3,t]/I_{\text{hom}}) \quad \text{and}$$
$$\overline{X}_0^J = \operatorname{Proj}(\mathbb{C}[x_1,x_2,y_1,y_2,y_3,z_1,z_2,z_3,t]/J_{\text{hom}})$$

of X_0. Their diamonds are given in figure 1.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & & 16 & & ? & & ? & & 16 & & 0 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & & 1 & & ? & & ? & & 1 & & 0 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

FIGURE 1. The diamonds of two projectivisations $\overline{X}_0^I$ (left) and $\overline{X}_0^J$ (right) of the regular fibre corresponding to $H = H_0 = \operatorname{Diag}(1,-1,0)$.

Remark 2. The variety $\overline{X}_0^J$ is an irreducible component of $\overline{X}_0^I$. Indeed, we find that $I \subset J$ and that J is a prime ideal (whereas I is not). The discrepancy of values in the middle row is corroborated by the discrepancy between the expected Euler characteristics of the compactifications.

Remark 3. Macaulay2 greatly facilitates cohomological calculations that are unfeasible by hand. However, the memory requirements rise steeply with the dimension of the variety. The unknown entries in our diamonds (marked with a '?') exhausted the 48GB of RAM of the computers of our collaborators at IACS Kolkata, without producing an answer.

Question. This leaves us with the open question of characterising all the compactifications of a given orbit produced by the method of homogenising the defining ideals.

Nevertheless, once again methods of Lie theory provide us with sharper tools, and we obtain a compactification that is natural from the Lie theory viewpoint.

Let w_0 be the principal involution of the Weyl group $\mathscr{W}$, that is, the element of highest length as a product of simple roots. For a subset $\Theta \subset \Sigma$ we put $\Theta^* = -w_0\Theta$ and refer to $\mathbb{F}_{\Theta^*}$ as the flag manifold dual to $\mathbb{F}_\Theta$. If H_Θ is a characteristic element for Θ then $-w_0 H_\Theta$ is characteristic for Θ^*. Then the diagonal action of G on the product $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$ as $(g,(x,y)) \mapsto (gx, gy)$, $g \in G$, $x, y \in \mathbb{F}$ has just one open and dense orbit which is G/Z_Θ.

Let x_0 be the origin of $\mathbb{F}_\Theta$. Since G acts transitively on $\mathbb{F}_\Theta$, all the G-orbits of the diagonal action have the form $G \cdot (x_0, y)$, with $y \in \mathbb{F}_{\Theta^*}$. Thus, the G-orbits are in bijection with the orbits through wy_0, $w \in \mathscr{W}$, where y_0 is the origin of $\mathbb{F}_{\Theta^*}$. We obtain:

Proposition 15. [GGSM2, Prop. 3.1] *The orbit $G \cdot (x_0, w_0 y_0)$ is open and dense in $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$ and identifies to G/Z_H.*

Remark 4. Katzarkov, Kontsevich, and Pantev [KKP] give three definitions of Hodge numbers for Landau–Ginzburg models and conjecture their equivalence. Understanding the relation between the diamonds we presented here and those Hodge numbers provides a new perspective to our work.

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ADDRESSES: CALLANDER, GRAMA, SAN MARTIN - IMECC - UNICAMP, DEPTO. DE MATEMÁTICA. RUA SÉRGIO BUARQUE DE HOLANDA, 651, CIDADE UNIVERSITÁRIA ZEFERINO VAZ. 13083-859 CAMPINAS - SP, BRASIL. GASPARIM - DEPTO. MATEMÁTICAS, UNIVERSIDAD CATÓLICA DEL NORTE, ANTOFAGASTA, CHILE. EMAILS: BRIANCALLANDER@GMAIL.COM, ETGASPARIM@GMAIL.COM, LINOGRAMA@GMAIL.COM, SMARTIN@IME.UNICAMP.BR.