ISOMORPHISMS OF MODULI SPACES

C. CASORRÁN AMILBURU, S. BARMEIER, B. CALLANDER, AND E. GASPARIM

ABSTRACT. We give infinitely many new isomorphisms between moduli spaces of bundles on local surfaces and on local Calabi–Yau threefolds.

CONTENTS

1. INTRODUCTION

To study moduli spaces of rank 2 bundles on local surfaces and local threefolds we present concrete descriptions of these moduli as quotients of the vector spaces of extensions of line bundles by holomorphic isomorphism. Our favourite varieties are the following:

$$Z_k := \mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-k)) \qquad \text{and} \qquad W_i := \mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(i-2) \oplus \mathscr{O}_{\mathbb{P}^1}(-i)),$$

together with moduli of bundles on them. Let ℓ denote the zero section of Z_k and denote by X_k the surface obtained from Z_k by contracting ℓ to a point; thus X_k is singular for $k > 1$. For a bundle E on a surface Z_k, let ℓ denote the zero section of $\mathscr{O}_{\mathbb{P}^1}(-k)$ considered as a subvariety of Z_k, and $\pi\colon Z_k \to X_k$ the map that contracts ℓ to a point x. Hence π is the inverse of blowing up x. In the following, we shall also let Y denote either W_i or Z_k.

Definition 1.1. The *charge* of a bundle $E \to Y$ around ℓ is the *local holomorphic Euler characteristic* of $\pi_* E$ at x, defined as

$$\chi\big(x, \pi_* E\big) := \chi\big(\ell, E\big) := h^0\big(X;\, (\pi_* E)^{\vee\vee}/\pi_* E\big) + \sum_{i=1}^{n-1} (-1)^{i-1} h^0\big(X;\, R^i \pi_* E\big)\,. \tag{1.1}$$

Note that we have only $\chi\big(\ell, E\big) = h^0\big(X;\, (\pi_* E)^{\vee\vee}/\pi_* E\big) + h^0\big(X;\, R^1 \pi_* E\big)$ since our spaces only have two coordinate charts (see 3.2).

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Definition 1.2. Let $\sim$ denote bundle isomorphism and introduce the following notation and definitions.

(1) $\mathcal{M}_{j_1,j_2}(Y) := \mathrm{Ext}^1(\mathcal{O}_Y(j_2),\mathcal{O}_Y(j_1)) \big/\!\sim$
(2) $\mathcal{M}_j(Y,0) := \mathcal{M}_{j,-j}(Y)$
(3) $\mathcal{M}_j(Y,1) := \mathcal{M}_{j+1,-j}(Y)$

Note that the second entry, that is either 1 or 0, denotes the *first Chern class* of the bundles considered in each case.
From such quotients we extract the following moduli spaces. Let $\epsilon = 0$ or 1.

(1) $\mathfrak{M}^1_j(Y,\epsilon) \subset \mathcal{M}_j(Y,\epsilon)$ consisting of elements given by an extension class vanishing to order exactly 1 over ℓ,
(2) $\mathfrak{M}^s_j(Y,\epsilon) \subset \mathfrak{M}^1_j(Y,\epsilon)$ consisting of elements with lowest charge χ_{low}, where $\chi_{\mathrm{low}} := \inf\{\chi(E) | E \in \mathfrak{M}^1_j(Y,\epsilon)\}$.

Remark 1.3. For W_1, it follows by lemma 2.1 that all rank 2 bundles are extensions of line bundles. In fact, we also have this filtrability for W_2 but not for W_i with $i \geq 3$.

Our main results are the following:

Theorem *(Coincidence of moduli of bundles on surfaces and threefolds)*
For all positive integers i, j, k, there are isomorphisms

$$\mathfrak{M}^1_{2j+\left\lfloor \frac{k-3}{2} \right\rfloor+\delta}(Z_k,\epsilon) \simeq \mathfrak{M}^1_j(W_i,\delta)$$

and birational equivalences

$$\mathfrak{M}^s_{2j+\left\lfloor \frac{k-3}{2} \right\rfloor+\delta}(Z_k,\epsilon) \dashrightarrow \mathfrak{M}^s_j(W_1,\delta)$$

when $\epsilon \equiv k+1 \bmod 2$ and $\delta \in \{0,1\}$.

Theorem *(Atiyah–Jones type statement for local moduli)*
For $q \leq 2(2j-k-2+\delta)$ there are isomorphisms

(ι) $H_q(\mathfrak{M}^1_j(Z_k),\delta) = H_q(\mathfrak{M}^1_{j+1}(Z_k),\delta)$
($\iota\iota$) $\pi_q(\mathfrak{M}^1_j(Z_k),\delta) = \pi_q(\mathfrak{M}^1_{j+1}(Z_k),\delta)$.

and for $q \leq 2(4j-3-2\delta)$ there are isomorphisms

($\iota\iota\iota$) $H_q(\mathfrak{M}^1_j(W_i),\delta) = H_q(\mathfrak{M}^1_{j+1}(W_i),\delta)$
(ιv) $\pi_q(\mathfrak{M}^1_j(W_i),\delta) = \pi_q(\mathfrak{M}^1_{j+1}(W_i),\delta)$.

Remark 1.4. We obtain isomorphisms between bundles E and F over Z_k with $c_1(F) = c_1(E)+2$ by tensoring with $\mathcal{O}(-1)$, as

$$\begin{pmatrix} z^{-j_1} & p \\ 0 & z^{-j_2} \end{pmatrix} \otimes z = \begin{pmatrix} z^{-j_1+1} & zp \\ 0 & z^{-j_2+1} \end{pmatrix}$$

so that we could consider $\epsilon \in \mathbb{Z}$, as long as $\epsilon \equiv k+1 \bmod 2$ still holds.

2. Filtrability and algebraicity

We deal with bundles on local surfaces and threefolds, that is, a neighborhood of a curve C embedded in a smooth surface or threefold W, typically the total space of a vector bundle N over C. We focus on the case when $C \simeq \mathbb{P}^1$. In the 2-dimensional case we focus on the case when N^* is ample, and in the 3-dimensional case we focus on Calabi–Yau threefolds.

Let W be a connected complex manifold (or smooth algebraic variety) and C a curve contained in W that is reduced, connected and a local complete intersection. Let $\widehat{C}$ denote the formal completion of C in W. Ampleness of the conormal bundle has a strong influence on the behaviour of bundles on $\widehat{C}$. We will use the following basic fact from formal geometry.

Lemma 2.1. [BKG2, thm. 3.2] *If the conormal bundle N_C^* is ample, then every vector bundle on $\widehat{C}$ is filtrable. If in addition C is smooth, then every holomorphic bundle on $\widehat{C}$ is algebraic.*

Remark 2.2. Ampleness of N_C^* is essential. For example, consider the Calabi–Yau threefold

$$W_i = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(i-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-i)).$$

Then W_1 satisfies the hypothesis of 2.1, hence holomorphic bundles on W_1 are filtrable and algebraic, whereas on W_2 filtrability still holds, but there exist proper holomorphic bundles W_2 that are not algebraic, and on W_i for $i \geq 3$ neither filtrability nor algebraicity hold, see [K] chapter 3.3.

3. Surfaces

We use the very concrete description of moduli spaces of rank 2 bundles over the surfaces $Z_k := \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ given in [BKG1]. Let ℓ denote the zero section inside Z_k. Given a bundle E over Z_k, its restriction to ℓ splits by Grothendieck's principle, and if $E|_\ell \simeq \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ then $(a_1, \ldots, a_r)$ is called the *splitting type* of E. By [Ga, thm. 3.3], a holomorphic bundle E over Z_k having splitting type (j_1, j_2) with $j_1 \leq j_2$ can be written as an algebraic extension

$$0 \longrightarrow \mathcal{O}(j_1) \longrightarrow E \longrightarrow \mathcal{O}(j_2) \longrightarrow 0 \tag{3.1}$$

and therefore corresponds to an *extension class*

$$p \in \mathrm{Ext}^1_{Z_k}(\mathcal{O}(j_2), \mathcal{O}(j_1)).$$

We fix once and for all coordinate charts on our surfaces $Z_k = U \cup V$, where

$$U = \mathbb{C}^2_{z,u} = \{(z,u) \in \mathbb{C}^2\} \qquad \text{and} \qquad V = \mathbb{C}^2_{\xi,v} = \{(\xi,v) \in \mathbb{C}^2\} \tag{3.2}$$

and

$$(\xi, v) = (z^{-1}, z^k u) \ \text{ on } \ U \cap V.$$

In these coordinates, the bundle E may be represented by a transition matrix in *canonical form* as

$$T = \begin{pmatrix} z^{-j_1} & p \\ 0 & z^{-j_2} \end{pmatrix}$$

where

$$p = \sum_{i=1}^{\lfloor (j_2-j_1-2)/k \rfloor} \sum_{l=ki+j_1+1}^{j_2-1} p_{il} z^l u^i \ . \tag{3.3}$$

Since we are interested in isomorphism classes of vector bundles rather than extension classes, we use the following moduli:

$$\mathcal{M}_{j_1,j_2}(Z_k) = \mathrm{Ext}^1(\mathcal{O}_{Z_k}(j_2), \mathcal{O}_{Z_k}(j_1)) \big/ \sim$$

where $\sim$ denotes bundle isomorphism. We observe that this quotient gives rise to a moduli stack, but we will only describe here subsets of its coarse moduli space considered as a variety. Considered just as a topological space, the full quotient will not be Hausdorff except in the trivial case, when it contains only a point. The latter happens when the only bundle with splitting type (j_1, j_2) is $\mathcal{O}_{Z_k}(j_1) \oplus \mathcal{O}_{Z_k}(j_2)$, that is, whenever $j_2 - j_1 < k+2$.

To specify the topology in this quotient space, we use the canonical form of the extension class (3.3). Then the coefficients of p written in lexicographical order form a vector in $\mathbb{C}^m$, where m is the number of complex coefficients appearing in the expression of p. We define an equivalence relation in $\mathbb{C}^m$ by setting $p \sim p'$ if (j_1, j_2, p) and (j_1, j_2, p') define isomorphic bundles over Z_k, and give $\mathbb{C}^m/\sim$ the quotient topology. Now setting $n := \lfloor (j_2 - j_1 - 2)/k \rfloor$, we obtain a bijection

$$\begin{aligned} \phi\colon \mathcal{M}_{j_1,j_2}(Z_k) &\to \mathbb{C}^m \big/ \sim \ , \\ \begin{pmatrix} z^{-j_1} & p \\ 0 & z^{-j_2} \end{pmatrix} &\mapsto \left(p_{1,k+j_1+1}, \ldots, p_{n,j_2-1}\right) \end{aligned}$$

and give $\mathcal{M}_{j_1,j_2}(Z_k)$ the topology induced by this bijection.

Now observe that it is always the case that $p \sim \lambda p$ for any $\lambda \in \mathbb{C} - \{0\}$. The moduli space is then evidently non-Hausdorff, as the only open neighborhood of the split bundle is the entire moduli space. In the spirit of GIT one would like to extract nice moduli spaces out of these quotient spaces. Clearly the split bundle needs to be removed, but there is quite a bit more topological complexity.

3.1. **Vanishing c_1 case: moduli spaces.** For rank 2 bundles E over Z_k with $c_1(E) = 0$ there is a non-negative integer j such that $E|_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j)$ and we will say E has splitting type j. We denote by $\mathcal{M}_j$ the moduli of all bundles with this fixed splitting type (see Definition 1.2, item (2)):

$$\mathcal{M}_j(Z_k, 0) := \mathrm{Ext}^1(\mathcal{O}_{Z_k}(-j), \mathcal{O}_{Z_k}(j)) \big/ \sim .$$

We now recall some results about the topological structure of these spaces and their relation to instantons. These moduli spaces are stratified into Hausdorff components by local analytic invariants. Given a reflexive sheaf E over Z_k we set:

$$\mathbf{w}_k(E) := h^0((\pi_* E)^{\vee\vee} / \pi_* E), \qquad \mathbf{h}_k(E) := h^0(R^1 \pi_* E).$$

called the *width* and *height* or E, respectively.

Definition 3.1. $\chi(\ell, E) := \mathbf{w}_k(E) + \mathbf{h}_k(E)$ is called the *local holomorphic Euler characteristic* or *charge* of E.

We quote the following results to show the connection with mathematical physics

Theorem 3.2. [BKG1, cor. 5.5] *Correspondence with instantons. An* $\mathfrak{sl}(2,\mathbb{C})$*-bundle over* Z_k *represents an instanton if and only if its splitting type is a multiple of* k.

Theorem 3.3. [BKG1, thm. 4.15] *Stratifications. If* $j = nk$ *for some* $n \in \mathbb{N}$*, then the pair* $(\mathbf{h}_k, \mathbf{w}_k)$ *stratifies instanton moduli stacks* $\mathcal{M}_j(k)$ *into Hausdorff components.*

Remark 3.4. Let us note the following:

- χ alone is not fine enough to stratify the moduli spaces.
- Constructing such a stratification for the non-instanton case is an open problem.
- There are various ways to obtain moduli spaces inside the $\mathcal{M}_j$. One possible choice is to take the largest Hausdorff component as our moduli space. This will produce compact moduli, and we study this case in section 3.2. A second, more natural choice is to fix some numerical invariant, to which end the local holomorphic Euler characteristic presents itself as the most natural candidate.

3.2. **Vanishing c_1 case: first order deformations.**

Notation 3.5. Let $\mathfrak{M}^1_j(Z_k,0) \subset \mathcal{M}_j(Z_k)$ denote the subset which parametrizes isomorphism classes of bundles on Z_k consisting of isomorphism classes of nontrivial first order deformations of $\mathcal{O}(j) \oplus \mathcal{O}(-j)$, that is, bundles E fitting into an exact sequence

$$0 \to \mathcal{O}(-j) \to E \to \mathcal{O}(j) \to 0 \tag{3.4}$$

whose corresponding extension class vanishes to order exactly one on ℓ (note that this excludes the split bundle itself). In other words, $\mathcal{I}_\ell = \langle u \rangle$ on the u-chart and consider only extensions $p \in \mathrm{Ext}(\mathcal{O}(j), \mathcal{O}(-j))$ with $p = uq$ and $u \nmid q$.

Remark 3.6. If $2j-2 < k$ then $\mathcal{M}_j(Z_k)$ consists of just a point represented by the split bundle, consequently if $2j-2<k$ then $\mathfrak{M}^1_j(Z_k,0) = \emptyset$.

A simple observation, which we now describe, then implies that $\mathfrak{M}^1_j(Z_k)$ is compact and smooth.

Theorem 3.7. [BKG1, thm. 4.9] *On the first infinitesimal neighbourhood, two bundles* $E^{(1)}$ *and* $F^{(1)}$ *with respective transition matrices*

$$\begin{pmatrix} z^j & p_1 \\ 0 & z^{-j} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} z^j & q_1 \\ 0 & z^{-j} \end{pmatrix}$$

are isomorphic if and only if $q_1 = \lambda p_1$ *for some* $\lambda \in \mathbb{C} - \{0\}$.

Remark 3.8. Note that no similar result holds true if we include higher order deformations, because then there are further identifications and the quotient space is no longer Hausdorff.

Corollary 3.9. $\mathfrak{M}^1_j(Z_k,0) \simeq \mathbb{P}^{2j-k-2}$.

3.3. **Vanishing c_1 case: minimal charge.** Another possible choice of moduli space, more compatible with the physics motivation, is to consider the subset of bundles on $\mathfrak{M}^1_j(Z_k,0)$ having fixed charge; this is preferable, because the charge is an analytic invariant on the bundles, and minimal charge corresponds to a generic choice for the corresponding instanton interpretation. In this case we take the open subset of the moduli of first order deformations defined by:

$$\mathfrak{M}^s_j(Z_k,0) := \{E \in \mathfrak{M}^1_j(Z_k,0) : \chi(E) = \chi_{\min}(Z_k)\}.$$

Charge is lower semi-continuous on the splitting type, and we have that the locus of bundles with charge higher than $\chi_{\min}$ is Zariski closed; in fact, such locus is determined by $k+1$ polynomial equations [BKG1, thm. 4.11].

Corollary 3.10. *$\mathfrak{M}^s_j(Z_k,0)$ is a quasi-projective variety, whose complement in $\mathbb{P}^{2j-k-2}$ is cut out by $k+1$ equations.*

Proof. On the first infinitesimal neighbourhood p_1 has $2j-k-1$ coefficients modulo projectivisation (see equation 3.3) and then, by means of Theorem 3.7, we arrive at the desired result. □

3.4. **Case $c_1 = 1$.** From expression (3.3) we can read off the case $c_1 = 1$ by setting $j_1 = -j$ and $j_2 = j+1$, considering extensions $\operatorname{Ext}^1_{Z_k}(\mathcal{O}(j+1),\mathcal{O}(-j))$. The form of the extension class restricted to the first infinitesimal neighborhood expressed in canonical coordinates is

$$\sum_{l=k-j+1}^{j} p_{1l}\, z^l u.$$

The coefficients vary in $\mathbb{C}^{2j-k}$, so that modulo the relation $p \sim \lambda p'$ we have:

Lemma 3.11. $\mathfrak{M}^1_j(Z_k,1) \simeq \mathbb{P}^{2j-k-1}$.

Proof. The proof of this lemma is just a modification of the proof of Theorem 3.7, which goes through successfully by replacing the appropriate js with $j+1$: On the first infinitesimal neighbourhood, two bundles $E^{(1)}$ and $F^{(1)}$ with respective transition matrices

$$\begin{pmatrix} z^j & p_1 \\ 0 & z^{-j-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z^j & q_1 \\ 0 & z^{-j-1} \end{pmatrix}$$

are isomorphic if and only if $q_1 = \lambda p_1$ for some $\lambda \in \mathbb{C} - \{0\}$. Thus, projectivising the space of bundles on the first formal neighbourhood gives the isomorphism classes in the case $c_1 = 1$ just like we had in the vanishing c_1 case. □

The moduli space $\mathfrak{M}^s_j(Z_k,1)$ of bundles with minimal charge can also be considered as well. Since charge is lower semi-continuous, the set $\mathfrak{M}^s_j(Z_k,1)$ of bundles in $\mathfrak{M}^1_j(Z_k,1)$ achieving minimal charge is Zariski open.

4. Threefolds

Consider the threefolds

$$W_i = \operatorname{Tot}(\mathcal{O}_{\mathbb{P}^1}(i-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-i))$$

to which we alluded earlier in section 2, and denote by ℓ the zero section inside W_i. We focus on the cases of rank 2 and either $c_1 = 0$ or else $c_1 = 1$ as we did in section 3 and for a bundle E over W_i such that $E|_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j)$ we call the non-negative integer j the splitting type of E. Note that here again $\operatorname{Pic} W_i \simeq \operatorname{Pic} \ell$ so we can avoid a subscript in the notation $\mathcal{O}(j)$.

We now consider only *algebraic extensions* over the W_i and then define moduli spaces analogous to the ones we defined in section 3. First the set of isomorphism classes of bundles with fixed splitting type:

$$\mathscr{M}_j(W_i) = \left\{E \to W_i : E|_\ell \simeq \mathscr{O}(j) \oplus \mathscr{O}(-j)\right\} / \sim ,$$

and

$$\mathfrak{M}^1_j(Z_k) \subset \mathscr{M}_j(Z_k)$$

the subset which parametrizes bundles on W_i which are nontrivial first order deformations of $\mathscr{O}(j) \oplus \mathscr{O}(-j)$, that is, bundles E fitting into an exact sequence

$$0 \to \mathscr{O}(-j) \to E \to \mathscr{O}(j) \to 0$$

whose corresponding extension class vanishes to order exactly one on ℓ (note that this excludes the split bundle itself). In local canonical coordinate charts, we have

$$\begin{aligned} &W_i = U \cup V, \quad \text{with} \quad U = \mathbb{C}^3 = \{(z, u_1, u_2)\}, \quad V = \mathbb{C}^3 = \{(\xi, v_1, v_2)\} \\ &\text{and} \quad (\xi, v_1, v_2) = (z^{-1}, z^{2-i}u_1, z^i u_2) \quad \text{in} \quad U \cap V. \end{aligned} \tag{4.1}$$

Then on the U-chart $\mathscr{I}_\ell = \langle u_1, u_2\rangle$ and elements of $\mathfrak{M}^1_j$ are determined by extension classes $p \in \mathrm{Ext}(\mathscr{O}(j), \mathscr{O}(-j))$ with either $p = u_1 p'$ or else $p = u_2 p''$ and $u_1 \nmid p'p''$, $u_2 \nmid p'p''$.

Lemma 4.1. [GK, cor. 5.6] *We have an isomorphism of varieties*

$$\mathfrak{M}^1_j(W_i, 0) \simeq \mathbb{P}^{4j-5}.$$

Once again, fixing a numerical invariant seems to be a preferable choice (as suggested by the last item on Remark 3.4), so we define:

$$\mathfrak{M}^s_j(W_i, 0) := \{E \in \mathfrak{M}^1_j(W_i, 0) : \chi(E) = \chi_{\min}(W_i)\},$$

and this is a Zariski open subvariety of $\mathfrak{M}^1_j$.

Lemma 4.2. $\mathfrak{M}^1_j(W_i, 1) = \mathbb{P}^{4j-3}$.

Proof. In canonical coordinates, an extension of $\mathscr{O}(j+1)$ by $\mathscr{O}(-j)$ may be represented over W_i by the transition matrix:

$$T = \begin{pmatrix} z^j & p \\ 0 & z^{-j-1} \end{pmatrix}.$$

On the intersection $U \cap V = \mathbb{C} - \{0\} \times \mathbb{C}^2$ the holomorphic functions are of the

$$p = \sum_{t=-\infty}^{\infty} \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} p_{rst} z^r u_1^s u_2^t.$$

By changing coordinates one can show that it is equivalent to consider p as

$$\begin{aligned} &(p_{-j,0,0} z^{-j} + \cdots + p_{j-1,0,0} z^{j-1}) \\ &+ (p_{-j-i+2,1,0} z^{-j-i+2} + \cdots + p_{j-1,1,0} z^{j-1}) u_1 \\ &+ (p_{-j+i,0,1} z^{-j+i} + \cdots + p_{j-1,0,1} z^{j-1}) u_2 \\ &+ \text{higher-order terms.} \end{aligned}$$

Therefore, counting coefficients on the first infinitesimal neighbourhood gives $4j-2$ coefficients giving dimension $4j-3$ after projectivising.

□

Theorem 4.3. *For all positive integers i, j, k, there are isomorphisms*

$$\mathfrak{M}^1_{2j+\left\lfloor\frac{k-3}{2}\right\rfloor+\delta}(Z_k,\epsilon) \simeq \mathfrak{M}^1_j(W_1,\delta)$$

and birational equivalences

$$\mathfrak{M}^s_{2j+\left\lfloor\frac{k-3}{2}\right\rfloor+\delta}(Z_k,\epsilon) \dashrightarrow \mathfrak{M}^s_j(W_1,\delta)$$

when $\epsilon \equiv k+1 \bmod 2$ and $\delta \in \{0,1\}$.

Proof. By setting $j \mapsto 2j+\left\lfloor\frac{k-3}{2}\right\rfloor+\delta$ in Corollary 3.9, we obtain isomorphisms

$$\mathfrak{M}^1_{2j+\left\lfloor\frac{k-3}{2}\right\rfloor+\delta}(Z_k,0) \simeq \mathbb{P}^{4j-3-2\delta}$$

for k odd. Similarly, we can use lemma 3.11 to obtain isomorphisms

$$\mathfrak{M}^1_{2j+\left\lfloor\frac{k-3}{2}\right\rfloor+\delta}(Z_k,1) \simeq \mathbb{P}^{4j-3-2\delta}$$

for k even. The required isomorphisms to $\mathfrak{M}^1_j(W_1,\delta)$ then follow from lemmas 4.1 and 4.2 for $\delta=0,1$, respectively.

To find the birational equivalences, first note that we have

$$\mathfrak{M}^s_{2j+\left\lfloor\frac{k-3}{2}\right\rfloor+\delta}(Z_k,\epsilon) \subset \mathfrak{M}^1_{2j+\left\lfloor\frac{k-3}{2}\right\rfloor+\delta}(Z_k,\epsilon) \text{ and } \mathfrak{M}^s_j(W_i,\delta) \subset \mathfrak{M}^1_j(W_i,\delta)$$

by definition. Lemma 3.10 shows that $\mathfrak{M}^s_{2j+\left\lfloor\frac{k-3}{2}\right\rfloor+\delta}(Z_k,\epsilon)$ is a quasi-projective variety and we now show that $\mathfrak{M}^s_j(W_1,\delta)$ is also quasi-projective.

For any bundle on W_1, [BKG2, lem. 5.2] shows that the width is always $\mathbf{w}(E) = h^0\left((\pi_*E)^{\vee\vee}\big/\pi_*E\right) = 0$. Thus, fixed charge is equivalent to fixed height. Since height is minimal on a Zariski open set of W_1 of codimension at least 3 given by the vanishing of certain coefficients of p, $\mathfrak{M}^s_j(W_1)$ is Zariski open in $\mathfrak{M}^1_j(W_1)$.

Restricting the isomorphisms above to a suitably small neighbourhood of these quasi-projective varieties then gives the required birational equivalences. □

Question 4.4. Since $\ell \subset W_i$ cannot be contracted to a point for $i>1$, our definition of charge does not apply. Can similar numerical invariants be defined for bundles on W_i, $i>1$? Some such invariants were defined in [K] chapter 3.5, though much remains to be understood about their geometrical meaning.

Theorem 4.5. *For $q \le 2(2j-k-2+\delta)$ there are isomorphisms*

(ι) $H_q(\mathfrak{M}^1_j(Z_k),\delta) = H_q(\mathfrak{M}^1_{j+1}(Z_k),\delta)$

($\iota\iota$) $\pi_q(\mathfrak{M}^1_j(Z_k),\delta) = \pi_q(\mathfrak{M}^1_{j+1}(Z_k),\delta)$.

and for $q \le 2(4j-3-2\delta)$ there are isomorphisms

($\iota\iota\iota$) $H_q(\mathfrak{M}^1_j(W_i),\delta) = H_q(\mathfrak{M}^1_{j+1}(W_i),\delta)$

($\iota\nu$) $\pi_q(\mathfrak{M}^1_j(W_i),\delta) = \pi_q(\mathfrak{M}^1_{j+1}(W_i),\delta)$.

Proof. The statements follow immediately from corollary 3.9 and lemmas 3.11, 4.1 and 4.2. □

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ELIZABETH GASPARIM AND BRIAN CALLANDER

IMECC – UNICAMP, RUA SÉRGIO BUARQUE DE HOLANDA, 651, CIDADE UNIVERSITÁRIA ZEFERINO VAZ, DISTR. BARÃO GERALDO, CAMPINAS SP, BRASIL 13083-859
E-mail address: gasparim@ime.unicamp.br
E-mail address: briancallander@googlemail.com

CARLOS CASORRÁN AMILBURU, DEPTO. DE ESTADÍSTICA E INVESTIGACIÓN OPERATIVA, UNIVERSIDAD DE ALICANTE,03080-ALICANTE ESPAÑA
E-mail address: casorranamilburu@msn.com

SEVERIN BARMEIER, GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO, TOKYO, 153-8914 JAPAN
E-mail address: s.barmeier@googlemail.com