Introduction to Geometric Langlands over $\mathbb{C}$

Lectures by Constantin typed by Elizabeth - 2007

The “classical” geometric Langlands correspondence is a package of dualities between structures associated to a compact *Riemannian surface* Σ and a (complex, semi-simple) *Lie group* G. The Langlands dual group ${}^LG$ appears on the other side of the duality. For $G = GL(1) = \mathbb{C}^*$ or more generally for $GL(n, \mathbb{C})$ the Langlands dual group is G itself. But even then, and even in the symmetric version of the correspondence, the predicted duality is far from trivial (a self-duality on a set or a category which is NOT the identity. The original motivation came from arithmetic representation theory (study of complex representations of groups $G(\mathbb{Q}_p)$ of G(number field) which has a consistent behaviour across all primes p). The translation is

number fields	$\Rightarrow$	Riemannian surfaces
(Dedekind domain, 1 dim ring)		(local ring, 1 dim)
$G(\mathbb{Q}_p)$	$\Rightarrow$	Loop groups (Kac–Moody groups)

but the naive representation theory connection breaks in this translation. It is thus remarkable that a reformulation of the LHS (Laumon; Drinfeld) makes sense, and seems to be true over $\mathbb{C}$.

Note: There is also a version when the Riemannian surface (= projective curve over $\mathbb{C}$) gets replaced by a projective curve over a finite field. For $GL(n)$ this was proved by Lafforgue. Frenkel and Gaitsgory are working on a “categorical translation” of geometric Langlands that would restore the link with Kac – Moody groups.

The geometric motivation for Langlands duality over $\mathbb{C}$ came from a different and independent direction, initiated by N. Hitchin, who studied the moduli spaces of holomorphic vector bundles over a Riemann surface and the complex-analytic geometry of their cotangent bundles (T^*M_G). Soon it was realised (Beilinson – Drinfeld) that Hitchin’s construction (*The Hitchin System*) was the underlying semi-classical system of a possible formulation of the geometric Langlands correspondence. That is, the geometric Langlands correspondence was the non-commutative deformation of a conjectural equivalence of derived categories of coherent sheaves between T^*M_G and $T^*M_{^LG}$. Subsequently they used these ideas to prove a small (but nontrivial and convincing) part of the geometric Langlands correspondence.

For $G = GL(1)$ or a torus (= product of $GL(1)$'s) the entire project has been beautifully solved using the *Fourier-Mukai* correspondence. The statements and the geometry are crystal clear but not trivial. The project was completed by a theorem of Polishchuck – Rothstein. We then have:

Classical Langlands

$$\begin{array}{rcl} D(Coh(T^*M_{GL(1)})) & \longleftrightarrow & D(Coh(T^*M_{GL(1)})) \\ \textit{Hecke eigensheaves} & \longleftrightarrow & \textit{points} \end{array}$$

Geometric Langlands

$$\begin{array}{rcl} D(\mathcal{D} - \textit{modules on } M_{GL(1)}) & \longleftrightarrow & Coh(GL(1) \textit{ local systems}) \\ \textit{Hecke eigensheaves} & \longleftrightarrow & \textit{points} \end{array}$$

Double deformed Langlands

$$\begin{array}{rcl} D(\mathcal{D}_h - \textit{modules on } M_{GL(1)}) & \longleftrightarrow & D(\mathcal{D}_{1/h} - \textit{modules on } M_{GL(1)}) \\ \textit{no points} & \textit{in} & \textit{sight} \end{array}$$

1 Geometric Langlands I

Hitchin's system gives a "generic Abelianization" of the underlying geometric picture for arbitrary G, of the form that was later recognised as an instance of t-duality in mirror symmetry. This provides another motivation for studying geometric Langlands; as a non-trivial but (perhaps) exactly solvable example of t-duality.

The work of Witten, Kapustin and Gukov gives a complete dictionary of ingredients between geometric Langlands and Mirror Symmetry (relating them to duality of QFT's in 4 dimensions).

We'll study the geometric ingredients in geometric Langlands (classical) and their noncommutative deformations. These are:

- holomorphic line bundles and Jacobians
- coherent sheaves and their derived categories

- Fourier–Mukai transform for Abelian varieties
- noncommutative deformations,$\mathcal{D}$-modules and twisted $\mathcal{D}$-modules.
- vector bundles on Riemann surfaces and Hitchin systems
- Hecke correspondence
- geometric classical Langlands via Fourier–Mukai
- Beillinson-Drinfeld construction of certain eigensheaves.

1.1 The Fourier Transform

The Fourier transform of a function $f\colon \mathbb{R} \to \mathbb{C}$ is by definition

$$\tilde{f}(k) := \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

and we have Fourier's inversion formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{-ikx} dk.$$

Here we observe that the Fourier transform is almost its own inverse. This as well as several other properties of classical Fourier transform will have analogues in the Fourier–Mukai transform. We recall some of these properties.

- interchanges local and global behaviour:

$$\begin{array}{rcl} \textit{rapid decay of } f & \leftrightarrow & \textit{smoothness of } \tilde{f} \\ f \textit{ polynomial} & \leftrightarrow & \tilde{f} \textit{ distribution supported at } 0 \\ f \equiv 1 & \leftrightarrow & \tilde{f} = 2\pi\delta_0(k) \end{array}$$

- takes Gaussians to Gaussians:

$$e^{-x^2/2a} \mapsto \sqrt{2\pi a} e^{-ak^2/2}$$

- takes multiplication to convolution:

$$\begin{array}{rcl} \alpha - eigenvalue\ for\ multiplication & = & \delta\ function\ at\ \alpha \\ & \Downarrow & (FourierTransform) \\ i\alpha - eigenvalue\ for\ convolution & = & Fourier\ mode\ e^{i\alpha k} \end{array}$$

E.g.: $(\delta_0' \star \varphi)(k) = \int \delta_0'(l)\varphi(k-l)dl = \int \delta_0(l)\varphi'(k-l)dl = \varphi'(k)$

$$e^{i\alpha k} \mapsto i\alpha e^{i\alpha}k$$

1.2 Invariant formulation of the Fourier transform

Let V be a vector space and V^* its dual, chose volume forms on both. We can define a Fourier transform between functions on V and V^*.

$$FT\colon \Lambda^\bullet V^* \to \Lambda^\bullet V$$
$$\varphi \mapsto \tilde{\varphi}(w) := \int_V \varphi(v) e^{\sum_k w_k \wedge v_k}$$

Physicist's trick: define an odd vector space V by making liner functions on V anti-commute. We now see the concept of odd Fourier transform.

Let $L \subset V$ be a lattice and $L^* \subset V^*$ be the dual lattice. Then the simplest case of Langlands duality considers $T = V/L$ and the dual torus $T^\vee = V^*/L^*$.

Note: $\pi_1 T \simeq L \simeq Hom(T^\vee, S^1) = Irrep(T^\vee)$ and
$\pi_1 T^\vee \simeq L^* \simeq Hom(T, S^1) = Irrep(T)$.

Proposition 1.1 *$H^*(T,\mathbb{R}) \simeq \Lambda^\bullet V^*$ and $H^*(T^\vee,\mathbb{R}) \simeq \Lambda^\bullet V$. The Fourier transform is Poincaré duality on T (on $T^\vee$) with respect to the chosen volume form.*

Recall that Poincaré duality gives a pairing

$$H^k(M) \otimes H^{dimM-k}(M) \to \mathbb{R}$$

$$\varphi \otimes \psi \mapsto \int_M \varphi \wedge \psi.$$

If M is compact, oriented, this pairing is non-degenerate and gives an isomorphism. In case of a torus we get the previous pairing. There is a more geometric construction of the duality pairing, given by the (cohomological) *Fourier–Mukai transform.*

Consider the kernel $\exp(\sum_k w_k \wedge v_k) \in H^*(T \times T^\vee)$. and define the map $H^*(T) \to H^*(T)$ by

$$\varphi \mapsto \int_T \varphi \wedge \exp(\sum_k w_k \wedge v_k) \in H^*(T^\vee).$$

Given the diagram

$$\begin{array}{ccc} & T \times T^\vee & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ T & & T^\vee \end{array}$$

then the Fourier–Mukai correspondence can be written as

$$\varphi \mapsto (\pi_2)_*(\pi_1^*\varphi \wedge K),$$

where $K$is the kernel. This version makes sense in much more general settings, even when we do not have non-degeneracy of the Poincaré duality pairing. It suffices to have the concepts of inverse images, direct images and products.

1.3 Bundles, connections

To make a geometric description of the Fourier–Mukai kernel we must discuss line bundles (unitary and holomorphic), Abelian varieties, coherent sheaves, and K-theory.

Definition 1.2 A complex *vector bundle* over X is a triple $V \xrightarrow{p} X$ where V and X are topological (resp. smooth manifolds, complex analytic) spaces, p is a continuous (resp. smooth, holomorphic) map, such that every point $x \in X$ has a neighborhood U with $p^{-1}(U) \simeq U \times \mathbb{C}^n$.

The vector space $p^{-1}(x)$ is called the *fibre* over x. If $n = 1$ then V is called a *line* bundle.

Example 1.3 The tangent bundle of a smooth (holomorphic) manifold is a topological (holomorphic) vector bundle.

Definition 1.4 A *section* of $V \xrightarrow{p} X$ over $U \subset X$ is a map $\sigma: U \to V$ satisfying $p \circ \sigma = id$. Sections over U form a module over functions on U.

A *connection* on a vector bundle is a rule for differentiating sections by first order differential operators on the base, which satisfy Leibniz rule:

$$D_t(f \cdot \sigma) = \partial_t f \cdot \sigma + f \cdot D_t(\sigma)$$

for any vector t tangent to X at x. D_t should be linear in t.

Example 1.5 V= the product bundle $X \times \mathbb{C}^n$, α = matrix valued 1-form on X. Set

$$D_t(\sigma) = \partial_t \sigma + \alpha(t) \cdot \sigma.$$

Definition 1.6 A connection is *flat* it is defines an action of the algebra of differential operators on X over the sections of V.

Remark 1.7 *On every open set U the algebra of differential operators is called $\mathcal{D}(U)$ and a flat vector bundle is the simplest example of a $\mathcal{D}$-module.*

Example 1.8 With $D = \partial + \alpha$ as before choose coordinates x_i so that $\alpha = \sum_i \alpha_i dx_i$. Then the obstruction to flatness is the curvature of D, $\partial\alpha_i/\partial x_j - \partial\alpha_j/\partial x_i + [\alpha_j, \alpha_i]$, for each pair of indices i, j.

Theorem 1.9 *(Monodromy) Choose a point $x \in X$ and let V_x be the fibre of V at x. Up to isomorphism, a flat vector bundle is determined by its monodromy representation $m: \pi_1(X) \to GL(V_x)$ (up to GL conjugation).*

Definition 1.10 The flat bundle is called *unitary* if for every x the image of $\pi_1(X)$ lies on the unitary subgroup of $GL(V_x)$ for a specified inner product, in that case, we can specify a metric on the fibers for which parallel transport is unitary.

Theorem 1.11 *Complex line bundles are determined up to topological isomorphism by their first Chern class in $H^2(X, \mathbb{Z})$. If X has no homology torsion, flat line bundles are topologically trivial.*

1.4 The Poincaré bundle

Since the torus $T = V/L$ has no homology torsion,, flat line bundle on it are topologically trivial, so flat connections have the form $D_i = \partial/\partial v_i + \alpha_i$ (derivative in the i-th direction) where $\alpha = \sum \alpha_i w^i$ is a 1-form on T. The monodromy representation of the connection, $\pi_1(T) \to GL_1(\mathbb{C}) = \mathbb{C}^*$ sends a cycle $C \subset T$ to $\exp \int_c \alpha$. This depends only on the image by exp of the cohomology class of α in $H^1(T, \mathbb{C}^*)$. So, up to isomorphism, flat connections are parametrised by $T^\vee$.

Corollary 1.12 *$T^\vee$ is the moduli space of flat unitary bundles on T.*

Consequently, we can construct a universal line bundle on $T \times T^\vee$, trivial along $\{0\} \times T^\vee$ and restricting on $T \times \{t^*\}$ to the flat line bundle with holonomy t^*. This universal line bundle $\mathcal{P}$ is called the Poincaré line bundle. It carries a flat connection along T.

Construction of $\mathcal{P}$: Start with the trivial bundle over $T \times V^*$, with the standard flat connection along V^*, and with connection $\partial - 2\pi i w$ along T at $T \times \{w\}$ (having monodromy $\exp(2\pi i \int_C w$ along C). The curvature $(\partial_k \alpha_l - \partial_l \alpha_k)$ of this connection is $2\pi i \sum_k v_k \wedge w_k$, so the Chern class $\sum_k v_k w_k$ is in the kernel of the cohomological Fourier–Mukai transform.

Observe that L^* acts on V^* by translations. We can lift this to an action on the total space of the line bundle by

$$\ell^* \cdot (t, w, \lambda) = (t, w + T^\vee, \lambda \cdot e^{2\pi i \ell^*(t)}).$$

Dividing out by L^* gives a line bundle with connection on $T \times T^\vee$. The connection is flat along T and along $T^\vee$, and the Chern class is $\sum v_k \wedge w_k$.

Remark 1.13 The monodromy representation along $T^\vee$ at $\{t\} \times T^\vee$ is $\ell \mapsto e^{2\pi i \ell^*(t)}$. So, $\mathcal{P}$ is the conjugate of the $T^\vee$-Poincaré line bundle on $T^\vee \times T$, where we are viewing T and the moduli of flat line bundles on $T^\vee$.

Remark 1.14 Integration in cohomology has a counterpart in K-theory, when $\mathcal{P} \mapsto \exp(\sum_k v_k \wedge w_k)$. We can define the Fourier–Mukai transform by

$$K^*(T) \to K^*(T^\vee)$$

$$E \mapsto (p_2)_! p_1^*(E \otimes \mathcal{P}).$$

2 Geometric Langlands II

2.1 Complex structures on tori

If V is a complex vector space, then $T = V/L$ is a complex compact manifold, which is also an Abelian group. If in addition T is algebraic, then it is called an Abelian variety. The dual Abelian variety is $T^\vee = V^\vee/L^\vee$ with the complex conjugate structure. This makes the Poincaré bundle holomorphic. $\mathcal{P}$ is then the universal line bundle over $T \times T^\vee$.

2.2 Dolbeault cohomology

The Dolbeault cohomology allows us to define the direct image (=push forward) of vector bundles in the holomorphic category. It leads not to vector bundles, but to coherent sheaves, or more precisely, to objects in the derived category of those. From a holomorphic vector bundle $E \to X$ one obtains the sheaf of holomorphic sections $\mathcal{O}(E)$, which to an open set $U \subset X$ assigns the module $\mathcal{O}(E)(U)$ of holomorphic sections of E over U.

Fact: The Dolbeaut operator is half of the de Rham operator, and can be defined on C^∞ sections of E from the holomorphic structure alone. Locally a C^∞ section is $\sum f_i s_s$ where the s_i form a local frame of holomorphic sections and the f_i are C^∞ functions. On functions, we have:

$$\bar{\partial}\colon C^\infty \to \Omega^{0,1}$$

$$f \mapsto \sum \frac{\partial f}{\partial \bar{z}^k} \cdot d\bar{z}^k,$$

where, corresponding to $z^k = x^k + iy^k$ we have

$$\frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial x^k}\right)$$

$$d\bar{z}^k = dx^k - idy^k.$$

The $\bar{\partial}$ operator gives rise to a complex

$$\Omega^{0,0} \xrightarrow{\bar{\partial}} \Omega^{0,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{0,dim\,X}$$

which locally resolves the subsheaf $\mathcal{O} \subset C^\infty$ of holomorphic functions. We can define $\bar{\partial}$ on $C^\infty(E)$ by $\bar{\partial}(\sum f_i s_i) = \sum \bar{\partial} f_i \cdot s_i$. This turns out to be well defined, and independent of the decomposition of $s = \sum f_i s_i$ in a holomorphic frame.

Definition 2.1 $H^q(X, E) := \dfrac{\ker \bar{\partial} \colon \Omega^{0,q}(E) \to \Omega^{0,q+1}(E)}{\mathrm{im}\bar{\partial} \colon \Omega^{0,q-1}(E) \to \Omega^{0,q}(E)}$ is the *q-th Dolbeault cohomology group* of E.

Theorem 2.2 *If X is compact, then $H^q(X, E)$ is finite dimensional.*

Let now $X \xrightarrow{p} Y$ be a fiber bundle. Then we can define a vertical Dolbeault complex of $\mathcal{O}(X)$-modules over Y. Grauert's direct image theorem asserts that, if p is proper, then the cohomology sheaves of the associated complex are locally finitely generated $\mathcal{O}$-modules (= coherent sheaves).

2.3 Coherent sheaves and derived categories

MOTIVATION: We want a category that contains holomorphic vector bundles, as well as their inverse and direct images (pull-backs and push-forwards). The push-forward is implemented by coherent sheaf cohomology along the fibers of a morphism, e.g. Dolbeault cohomology, which results in sheaves, not bundles.

Definition 2.3 A *coherent sheaf* over a complex manifold is a sheaf of Abelian groups which is locally the cokernel of a morphism $\mathcal{O}^{\oplus m} \xrightarrow{\varphi} \mathcal{O}^{\oplus n}$.

Note that φ is an $\mathcal{O}$-module map, locally given by and $n \times m$ matrix of holomorphic functions.

Remark 2.4 Coherent sheaves are $\mathcal{O}$-modules, precisely the locally finitely presented $\mathcal{O}$-modules.

Example 2.5 over $\mathbb{C}$ the cokernel of the multiplication by z gives the short exact sequence

$$\mathcal{O} \xrightarrow{z} \mathcal{O} \to \mathbb{C}_0$$

with cokernel a *skyscraper sheaf* $\mathbb{C}_0$ supported at zero.

NOTE: Something funny happens here. The kernel of the morphism above is 0, but the fiberwise kernel is $\mathbb{C}$! Homological algebra comes to the rescue: $Tor_1^{\mathcal{O}}(\mathbb{C}_0, \mathbb{C}_0) = \mathbb{C}_0$ recovers the missing $\mathbb{C}_0$ if we need it.

NOTE: Starting with vector bundles, even f_* can lead to a sheaf and not a bundle. E.g. take $p\colon \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$ the blow up at a point, with exceptional divisor $= E$ then the direct image $p_*\mathcal{O}(-E)$ is $\mathfrak{m}_0$ where $\mathfrak{m}$ is the maximal ideal at zero in $\mathbb{C}^2$.

Theorem 2.6 *(Grauert's direct image theorem) $f\colon X \to Y$ a proper map between complex spaces, and $\mathcal{F} \to X$ coherent, then all the higher direct images $R^i f_*\mathcal{F}$ are coherent, and vanish for $i > \dim X$.*

FACT: Just coherent sheaves are not enough! We want also to keep track of the homological algebra information that is missing, such as in the previous example. Otherwise, we could only write the E^2 terms for spectral sequences, but would be missing the information required to compute differentials. The derived category contains precisely this missing information that goes into the connecting data.

We start with the honest theory of complexes of coherent sheaves, and then discard the information that will never appear in cohomology.

Definition 2.7 Let A be an Abelian category (e.g. coherent sheaves on a variety), and set

- $K^b(A)$ = category of bounded complexes of objects in A
- $K^+(A)$ = category of complexes of objects in A that are bounded below
- $K^-(A)$ = category of complexes of objects in A that are bounded above

in each of these categories, the morphisms are maps of complexes. A map of complexes that induces isomorphism in cohomology is called a *quasi-isomorphism*. Then we set:

- $D^b(A)$ = the category with the same objects as $K^b(A)$ but where we formally invert quasi-isomorphisms
- $D^+(A)$ = obtained from $K^+(A)$ formally inverting quasi-isomorphisms

- $D^-(A)$ = obtained from $K^-(A)$ formally inverting quasi-isomorphisms.

Thus, in the $D^\bullet$ categories, morphisms are formally finite chains:

$$\begin{tikzcd}[column sep=small] C \arrow[dr] & & C'' \arrow[dl, "q.i."'] \arrow[dr] & & C^{iv} \arrow[dl, "q.i."'] \\ & C' & & C''' & \end{tikzcd}$$

modulo an equivalence relation. But a bit of technical work cleans up the presentation (factor through the cohomology category, which admits a "calculus of fractions") and reduces the chain to a single zig-zag.

Example 2.8 If A is semi-simple, i.e. every short exact sequence splits (e.g. Vect) then $D(A)$ = category of graded objects in A, and homs are maps that preserve the grading.

Example 2.9 If A has homological dimension 1 (e.g. modules over a Dedekind domain), then every object in $D(A)$ is isomorphic to the direct sum of its cohomologies. But $D(A)$ is not equivalent to the category of graded A-objects.

In $D(\mathbb{Z} - modules)$ there is a morphism from $\mathbb{Z}/2$ viewed as a complex concentrated in degree 0 to $\mathbb{Z}$ in degree -1:

$$\begin{array}{ccc} {\scriptstyle -1} & & {\scriptstyle 0} \\ [\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}] \\ \downarrow^{Id} & & \downarrow^{0} \\ [\mathbb{Z}] & & 0 \end{array}$$

but there is no such morphism in graded $\mathbb{Z}$-modules.

Example 2.10 $Coh(\Sigma)$ where Σ is a Riemann surface. in $D(Coh(\Sigma)$ every object is isomorphic to a sum of graded a vector bundle and a graded skyscraper sheaf.

Example 2.11 In dimension ≥ 2 there are extra objects. For instance, the complex

$$\mathbb{C}[x,y]^{\oplus 2} \xrightarrow{\binom{x}{y}} \mathbb{C}[x,y]$$

has kernel $\mathbb{C}[x,y]$ mapping by $[y,-x]$ and cokernel $\mathbb{C}$ at 0, but is a non-trivial extension of the two. In fact, it is a Yoneda representative for the nontrivial element in $Ext^2(\mathbb{C}, \mathbb{C}[x,y]) = \mathbb{C}$.

Features of derived categories:

- do not have good concepts of kernels and cokernels (the standard definition will produce very few such objects). Instead, there are distinguished triangles, originating from short exact sequence of complexes: $X \to Y \to Z \to X[1]$
- have a shift functor [1]
- satisfy the axioms of a triangulated category
- ordinarily, half exact functors on Abelian categories tend to have derived analogues which are exact. Left exact functors (e.g. global sections of a coherent sheaf) have a right derived version (see below). Right exact functors (e.g. tensor) have left derived functors. (Some technical conditions apply.)

Features specific to coherent sheaves:

- On a smooth quasi-projective variety, every object in $D\,Coh$ is representable by a complex of vector bundles.
- The derived functor $\mathbf{R}\Gamma$ (global sections) of a complex of vector bundles is computed by the total Dolbeault complex.
- On a smooth algebraic variety, $D\,Coh$ is equivalent to the full subcategory of $D(\mathcal{O} - modules)$ consisting of objects whose cohomology sheaves are coherent.

Theorem 2.12 *For X, Y projective varieties, $\mathcal{F} \in D^{-}Coh(X \times Y)$, the assignment*

$$\mathcal{S} \mapsto \mathbf{R}\pi_{Y*}(\mathcal{F} \otimes^{\mathbf{L}} \pi_X^* \mathcal{S})$$

defines and exact functor $D^{-}Coh(X) \to D^{-}Coh(Y)$.

Denote this functor by $\phi_{\mathcal{F}}$. The image $\phi_{\mathcal{F}}(\mathcal{S}) = \mathcal{F} \star \mathcal{S}$ is called the convolution of $\mathcal{F}$ and $\mathcal{S}$.

Theorem 2.13 *(Orlov) If X and Y are smooth, then every exact equivalence $D^{-}Coh(X) \to D^{-}Coh(Y)$ is given by $\phi_{\mathcal{F}}$ for some $\mathcal{F} \in D^{-}Coh(X \times Y)$.*

Theorem 2.14 *If $\mathcal{F} \in D^bCoh(X \times Y)$ and $\mathcal{G} \in D^b(Coh(Y \times Z)$, then $\mathcal{G} \star \mathcal{F} \in D^bCoh(X \times Z)$ and $\phi_{\mathcal{G}} \circ \phi_{\mathcal{F}}$ is given by convolution with $\mathcal{G} \star \mathcal{F}$, that is, $\phi_{\mathcal{G}} \circ \phi_{\mathcal{F}} = \phi_{\mathcal{G}\star\mathcal{F}}$.*

Idea of Proof: Use base change

$$\begin{tikzcd}
 & & X \times Y \times Z \arrow[dl, "Q_Y"'] \arrow[dr, "P_Y"] & & \\
 & X \times Y \arrow[dl] \arrow[dr, "p_Y"] & & Y \times Z \arrow[dl, "q_Y"'] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

$q_Y^* \circ \mathbf{R}(p_Y)_* = \mathbf{R}(P_Y)_* \circ Q_Y^*$ for $Supp\mathcal{E}$ proper over Y.

Theorem 2.15 *(Mukai) Let A and $A^\vee$ be dual Abelian varieties of dimension n. Let $\mathcal{P}$ and $\mathcal{P}^\vee$ be the Poincaré bundles on $A \times A^\vee$ and $A^\vee \times A$. Then*

$$\begin{aligned}
\phi_{\mathcal{P}} \circ \phi_{\mathcal{P}^\vee} &= (-1_A)^*[-n] \\
\phi_{\mathcal{P}^\vee} \circ \phi_{\mathcal{P}} &= (-1_{A^\vee})^*[-n].
\end{aligned}$$

3 Geometric Langlands III

3.1 More on derived categories

Example 3.1 Let $Vect$ be the category of finite dimensional vector spaces over $\mathbb{C}$. In this category, every complex of vector spaces is quasi-isomorphic to its cohomology, and also any morphism on the derived category is determined by the induced map in cohomology. Indeed, one can check that here two maps of complexes inducing the same map in cohomology are homotopic, and homotopic morphisms become equal in the derived category. Consequently, denoting by $Gr(Vect)$ the category of graded vector spaces, we have:

$$\begin{array}{rcl}
D(Vect) & \simeq & Gr(Vect) \\
\text{complex with zero differential} & \leftrightarrow & \text{cohomology}
\end{array}$$

Example 3.2 Let $Mod-\mathbb{Z}$ denote the category of finitely generate Abelian groups. It is a fact that in this category every complex is quasi-isomorphic to its cohomology. However,

$$D(Mod-\mathbb{Z}) \not\simeq Gr(Mod-\mathbb{Z})$$

because the morphisms are different. In fact, in $Gr(Mod{-}\mathbb{Z})$, $Hom(\mathbb{Z}/p[-1], \mathbb{Z}) = 0$ for degree reasons. But in $D(Mod-\mathbb{Z})$ instead $Hom(\mathbb{Z}/p[-1], \mathbb{Z}) = \mathbb{Z}/p$, in fact, in $D(Mod-\mathbb{Z})$ it holds that $Hom(\mathbb{Z}/p, \mathbb{Z}) = Ext^1(\mathbb{Z}/p, \mathbb{Z})$ and, by the example in the previous lecture, there is morphism

$$\begin{array}{ccccc} 0 & & 1 & & \\ [\mathbb{Z} & \xrightarrow{p} & \mathbb{Z}] & \rightarrow & 0 \\ \downarrow & & \downarrow^{0} & & \\ [\mathbb{Z}] & & 0 & \rightarrow & 0 \end{array}$$

which is nonzero in $D(Mod-\mathbb{Z})$ because its dual is the surjection $\mathbb{Z} \rightarrow \mathbb{Z}/p$.

Example 3.3 If $dim\, R > 1$ then it is not true that $Ob(D(Mod-R))$ are isomorphic to their cohomology. See example 2.11.

A complex with a single non-zero cohomology group is quasi-isomorphic to its cohomology.

$$\begin{array}{ccccccccc} & \rightarrow & C^{p-1} & \rightarrow & C^p & \xrightarrow{\partial_p} & C^{p+1} & \rightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & \rightarrow & C^{p-1} & \rightarrow & Ker\, \partial_p & \rightarrow & 0 & \rightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & 0 & \rightarrow & H^p & \rightarrow & 0 & & \end{array}$$

Here the vertical arrows induce isomorphism in cohomology, hence are quasi-isomorphisms.

An object with cohomology H^p and H^q in 2 degrees only, $p < q$, is determined up to isomorphism by an element of $Ext^{q-p+1}(H^q, H^p)$. It is canonically isomorphic to a complex concentrated in the interval $[p, q]$. This gives a Yoneda representation for the extension class.

Remark 3.4 In $Mod{-}R$ you can compute $Ext^n(A, B)$ from a free resolution of A. Taking

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow A.$$

and applying $Hom(., B)$ gives

$$\cdots \leftarrow Hom(F_1, B) \rightarrow Hom(F_0, B)$$

then $Ext^n(A, B)$ is the $n-$th cohomology of this complex.

In general, an object in D is determined up to isomorphism its cohomology groups and a sequence of extension classes. The information about extension classes is concealed in the formalist of derived categories.

Analogy for Topologists:

derived categories	$\leftrightarrow$	homotopy category of topological space
quasi-isomorphism	$\leftrightarrow$	weak homotopy equivalence
cohomology groups	$\leftrightarrow$	homotopy groups
extension data	$\leftrightarrow$	k- invariants in the Postnikov tower of a space

3.2 Abelian categories and triangulated categories

In an Abelian category there exist the concepts of monomorphism, epimorphism, short exact sequence, kernel, cokernel, etc.

In a derived category the respective notions give nothing sensible: a quasi-isomorphism of complexes need not be represented by an injective or surjective map. Instead, in the derived category there is the notion of exact triangles $A \to B \to C \to A[1]$. Properties of exact triangles are axiomatized in the notion of triangulated categories. We'll just recall that an exact functor between derived categories is a $\mathbb{C}-$linear functor which commutes with the shift and takes exact triangles to exact triangles.

Example 3.5 The main case to consider is that half exact functors between Abelian categories often induce exact derived functors between derived categories. A particular such example is the functor of global sections $\Gamma\colon Coh(X) \to Vect$ defined by $\mathcal{S} \mapsto \Gamma(X, \mathcal{S})$. Γ is left exact and has a right derived functor $\mathbf{R}\Gamma\colon DCoh(X) \to DVect$.

More generally, if $f\colon X \to Y$ is proper, it induces $f_*\colon Coh(X) \to Coh(Y)$ and $\mathbf{R}f_*\colon DCoh(X) \to DCoh(Y)$, represented by the relative Dolbeault complex if f is a fiber bundle.

Remark 3.6 When X is quasi-projective, $DCoh(X)$ is equivalent to the full subcategory of sheaves of $\mathcal{O}-$modules on X with coherent cohomology sheaves. This is important because the Dolbeault complex is not a complex of coherent sheaves. If the conclusion of the previous statement fails, then the correct "derived category" is the second one.

Recall from theorem 2.12 that For any $\mathcal{E} \in DCoh(X \times Y)$ whose support is proper over Y,

$$\mathcal{F} \mapsto (\mathbf{R}p_Y)_* \circ (\phi\mathcal{E} \otimes^{\mathbf{L}} p_X^* \mathcal{F}) = \mathcal{E} \star \mathcal{F}$$

gives an exact functor $DCoh(X) \to DCoh(Y)$.

Remark 3.7 When the varieties are not smooth, p^* must be $\mathbf{L}p^*$ and then may have to be in D^-Coh (complexes bounded below).

Remark 3.8 When X is projective $\mathcal{E}$ is determined by $\phi_{\mathcal{E}}$, because the collection of $\phi_{\mathcal{E}}(L^{\otimes n})$ for large values of n determine $\mathcal{E}$.

Corollary 3.9 *$\phi_{\mathcal{E}}$ and $\phi_{\mathcal{F}}$ give inverse equivalences iff $\phi_{\mathcal{E}\star\mathcal{F}} = \Delta_* \mathcal{O}_Y \in Coh(Y \times Y)$ and $\phi_{\mathcal{F}\star\mathcal{E}} = \Delta_* \mathcal{O}_X \in Coh(X \times X)$.*

A result of Orlov can reduce the check to a "pointwise" computation.

Theorem 3.10 *(Orlov) X, Y smooth projective over $\mathbb{C}$. Then $\phi_{\mathcal{E}}$ is fully faithful iff the images $\phi_{\mathcal{E}}(\mathbb{C}_x)$ of skyscraper sheaves at $x \in X$ satisfy*

$$Hom(\phi_{\mathcal{E}}(\mathbb{C}_{x_1}), \phi_{\mathcal{E}}(\mathbb{C}_{x_2})) = 0, \ for \ x_1 \neq x_2$$

and $Ext(\phi_{\mathcal{E}}(\mathbb{C}_x), \phi_{\mathcal{E}}(\mathbb{C}_x))$ lives in degree $[1, \cdots, \dim X]$.

For Abelian varieties, $\phi_{\mathbb{C}_x}$ are flat line bundles on A^*. In this case we get that $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{P}^\vee}$ are fully faithful and they are adjoint functions, which implies the equivalence.

3.3 The Jacobian variety of a Riemann surface

Let J_0 denote the moduli space of isomorphism classes of flat unitary line bundles on Σ. There is a map

$$\mathbb{R}^{2g}/\mathbb{Z}^{2g} \equiv H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z}) \stackrel{\exp}{\to} J_0 = H^1(\Sigma, U(1)).$$

We have a complex structure on J_0 coming from its interpretation as moduli of holomorphic structures on the trivial line bundle. The $(0,1)$-part of any flat connection gives a $\bar{\partial}$-operator, hence, a holomorphic structure on the

line bundles. (A section σ is holomorphic if $\bar{\partial}\sigma = 0$.) We have $H^1(\Sigma, \mathbb{C}) = \{harmonic\ 1 - forms\} = H^{1,0}(\Sigma) \oplus H^{0,1}(\Sigma)$ by Hodge decomposition, and $H^1(\Sigma, \mathbb{C}) = H^0(\Sigma, \Omega^1) \oplus H^1(\Sigma, \mathcal{O})$. Notice that
1) Complex conjugation interchanges these spaces.
2) Poincaré duality gives a non-degenerate skew pairing

$$\Lambda^2 H^1(\Sigma, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z}) \equiv \mathbb{Z}$$

leading to an isomorphism

$$H^1(\Sigma, \mathcal{O}) \simeq H^0(\Sigma, \Omega^1)^\vee \simeq \overline{H^1(\Sigma, \mathcal{O})^\vee}.$$

The holomorphic structure on J_0 is defined to be $H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbb{Z})$ and is isomorphic to that on the dual Abelian variety $J_0^* = \overline{H^1(\Sigma, \mathbb{Z})}/H^1(\Sigma, \mathbb{Z})$ (self-duality of the degree 0 Jacobian).

Remark 3.11 More generally for a torus T ($\simeq U(1)^n$ but not canonically) and with complexification $T_{\mathbb{C}}$ there is a natural duality between the "Jacobian" of holomorphic $T_{\mathbb{C}}$-bundles and that of holomorphic $T_{\mathbb{C}}^\vee$- bundles, where $T_{\mathbb{C}}^\vee$ denotes the Langlands dual torus $T_{\mathbb{C}} = V/L$ and $T_{\mathbb{C}}^\vee = V^\vee/L^\vee$. This does not require a choice of basis in T.

Other holomorphic line bundles, are separated by degree in $H_2(\Sigma, \mathbb{Z}) = \mathbb{Z}$) giving

$$Pic(\Sigma) = \amalg_d Pic^d(\Sigma) \simeq \amalg_d J_0$$

and after a choice of a degree one line bundle L on Σ there is an isomorphism $L^{\otimes d} \otimes J_0 \simeq Pic^d$.
Problem: Including these seems to break Langlands duality, because for $T_{\mathbb{C}}$-bundles the components are labelled by $\pi_1 T_{\mathbb{C}}$ whereas for the dual $T_{\mathbb{C}}^\vee$ is labelled by $\pi_1 T_{\mathbb{C}}^\vee = Hom(T_{\mathbb{C}}, GL(1))$ and there is no identification of these two. This problem is solved by including automorphisms (passing from moduli spaces to moduli stacks).

Every holomorphic line bundles on Σ has a $GL(1)$ worth of automorphisms (scaling of a section). The correct "moduli object" is then $J \times BGL_1$ or in general $J(T) \times BT_{\mathbb{C}}$. But the factorization is not quite canonical.

Moral: coherent sheaves on J_0 are supposed to carry a fibrewise action of $GL(1)$ or of $T_{\mathbb{C}}$ in general. We can decompose the category of coherent sheaves according to $T_{\mathbb{C}}$ irreducibles. Hom between sheaves must preserve the $T_{\mathbb{C}}$ action, so this gives an orthogonal decomposition. Consequently, the relevant categories of coherent sheaves for the classical Langlands are

$$for\ T_{\mathbb{C}} : \Pi_{\ell \in \pi_1 T} \bigoplus_{\lambda \in (\pi_1 T)^\vee} DCoh(J(T))_\ell^\lambda$$

$$for\ T_{\mathbb{C}}^\vee : \Pi_{\lambda \in (\pi_1 T)^\vee} \bigoplus_{\ell \in \pi_1 T} DCoh(J(T^\vee))_\lambda^\ell$$

These are orthogonal decompositions of the categories, meaning that Hom between objects in different components are always zero. Fourier–Mukai exchanges the weight under the automorphism group with the component. E.g. coherent sheaves with compact support in both sides correspond exactly.

Note: some growth condition must be specified so that the two categories become equivalent. It is important that we do get an exact match because in the non-Abelian case the components get mangled up together into a single component, so no "correction by hand" can be done.

3.4 Convolution versus multiplication

The Fourier transforms exchanges "coherent states", that is, interchanges eigenvalues for differentiation ($e^{i\alpha x}$) with eigenvector for multiplication (δ functions). Fourier–Mukai interchanges flat line bundles with skyscraper sheaves. Note that skyscraper sheaves are eigensheaves for the operator of tensoring by a coherent sheaf, since

$$\mathcal{S} \otimes^{\mathbf{L}} \mathcal{O}_\alpha \simeq (derived\ fibre\ of\ \mathcal{S}\ at\ \alpha) \otimes \mathcal{O}_\alpha.$$

Proposition 3.12 *Any $\mathcal{F} \in DCoh(A)$ with*

$$\mathcal{S} \otimes^{\mathbf{L}} \mathcal{F} \simeq (derived\ fibre\ of\ \mathcal{S}\ at\ \alpha) \otimes \mathcal{F}$$

is equivalent to a sum of shifted copies of $\mathcal{O}_\alpha$.

Idea of proof: First show that $\mathcal{F}$ is supported at α because tensoring with $i_*\mathcal{O}_{A-\alpha}$ kills it. Then tensoring with a skyscraper sheaf at α leads to a direct sum of skyscraper sheaves.

Theorem 3.13 *For any $\mathcal{S} \in DCoh(A)$ and line bundle $L_{(\alpha)}$*

$$\mathcal{S} \star L_{(\alpha)} \simeq Ext(L_{(\alpha)}, \mathcal{S}) \otimes L_{(\alpha)}.$$

Moreover, any sheaf with this property is a a sum of shifted copies of the $L_{(\alpha)}$.

Idea of proof: Observe that $L_{(\alpha)}$ is isomorphic to all of its translates, by an isomorphism defined by the holonomy representation. So, given the multiplication $m: A \times A \to A$, the fibre at $a \in A$ is the set of all points $(x, a - x)$ and there is an isomorphism $\mathcal{S} \boxtimes L_{(\alpha)} \simeq L_{(\alpha^{-1})} \otimes \mathcal{S}$.

Theorem 3.14 *Fourier–Mukai transform takes multiplication to convolution shifted by $n = \dim A$.*

Proof: We need to show that for $\Delta: A \to A \times A$, and

$$\begin{array}{rcl} m: A \times A^\vee \times A \times A^\vee & \to & A \times A \times A^\vee \\ (a, \alpha, b, \beta) & \mapsto & (a, b, \alpha\beta) \end{array}$$

we have

$$\mathbf{R}m_*: (\mathcal{P} \boxtimes \mathcal{P}^\vee) = \Delta_* \mathcal{P}[n].$$

Then for $\mathcal{E}, \mathcal{F} \in DCoh(A)$ we have

$$(\mathcal{E} \otimes \mathcal{P}) \boxtimes (\mathcal{F} \otimes \mathcal{P}^\vee) \mapsto \mathcal{E} \otimes \mathcal{F} \otimes \mathcal{P}[-n].$$

The fibre of $\mathcal{P} \boxtimes \mathcal{P}^\vee$ over (a, b) is $L_{(a)} \boxtimes L_{(b)}$. But convolution with 0 if $a \neq b$ and is $L_{(a)}[-n]$ if $a = b$.

3.5 Hecke correspondence

In the case of a Jacobian, we can impose the eingensheaf condition using geometric information. Morally: an eigensheaf for the convolution is and eigensheaf for translations (=convolution by a skyscraper sheaves).

Let $p \in \Sigma$, and for any holomorphic line bundle L let $\mathcal{O}(L(p))$ be the sheaf of meromorphic sections of L having a single simple pole at p and holomorphic elsewhere.

Lemma 3.15 *$\mathcal{O}(L(p))$ is the sheaf of sections of a holomorphic line bundle with $deg = deg(L) + 1$, denoted $L(p)$. The induced map on Jacobians $J_p \to J_{p+1}$ is an isomorphism called the Hecke correspondence.*

Proof: Note that $z^{-1}\mathbb{C}[z]$ is a free $\mathbb{C}[z]$-module of rank 1, thus corresponding to a line bundle.

Remark 3.16 For $p, q \in \Sigma$, $L(p-q)$ is a point in the Jacobian of Σ. For a fixed q, the assignment $p \mapsto L(p-q)$ defines a holomorphic map $\varphi\colon \Sigma \to J_0$ called the *Abel–Jacobi* map.

Theorem 3.17 *Let φ be the Abel–Jacobi map:*
ι. φ is an embedding.
$\iota\iota$. $\varphi\colon H_1(\Sigma) \to H_1(J_0)$ is an isomorphism.
$\iota\iota\iota$. $im\,\varphi$ generates J_0.
$\iota\nu$. $\mathcal{P}$ on $J_0 \times J_0$ restricts to the universal bundle on Σ.

Remark 3.18 *More canonically, given $\Sigma \to J_0$, there is $S^g\Sigma \to J_g$ with image on Θ divisor. A choice of $\sqrt{K}$ of degree $g-1$ moves the Θ divisor back to J_0 (theory of Θ functions).*

Corollary 3.19 *Imposing the eigensheaf condition with respect to translations in J_0 is equivalent to imposing an eigensheaf condition with respect to $p-q$ for $p, q \in \Sigma$. Imposing the condition for a single point relates the different components of J.*

The *global Hecke eigensheaf condition* is then defined by:

$$\begin{array}{rcl} id \times H\colon \Sigma \times J_d & \to & \Sigma \times J_{d+1} \\ (p, L) & \mapsto & (p, L(p)) \end{array} .$$

More generally given a torus T we have Hecke maps labelled by components of $J(T)$

$$id \times H_\gamma\colon (p, \mathcal{T}) \mapsto (p, \mathcal{T}(p, \gamma),$$

where $\gamma \in \pi_1 T$ and $\mathcal{T}$ is a principal $T-$bundle.

Remark 3.20 For each weight $\lambda\colon T \to GL(1)$ of T we get an associated line bundle J_λ and the Hecke maps H_γ interchanges this line bundle with $J_\lambda(p \cdot \lambda(\gamma))$.

Definition 3.21 A *Hecke eigensheaf* $\mathcal{F} \in DCoh(J(T)\times BT)$ with eigenvalue $\mathcal{T} \in J(T^\vee)$ is an object with the property

$$(id \times H)_*: (\theta \boxtimes \mathcal{F}) = L_\gamma^{-1} \boxtimes \mathcal{F},$$

where L_γ^{-1} is the line bundle associated to $\mathcal{T}$ and to $\gamma: T^\vee \to GL(1)$.

Theorem 3.22 *The Hecke eigensheaves on $J(T) \times BT$ are the restrictions of the Poincaré bundle to points in $J(T^\vee)$.*

The labelling of components in $\pi_1 T^\vee = (\pi_1 T)^*$ becomes the weight of the T action.

4 Geometric Langlands IV

Recall that $\mathcal{F} \in DCoh(X)$ is an eigensheaf for the operation $\mathcal{S}\otimes$ if

$$\mathcal{S} \otimes^{\mathbf{L}} \mathcal{F} \simeq E(\mathcal{S}) \times \mathcal{F}$$

for some exact functor $E: DCoh(X) \to DVect$.

Note: This is a sensible requirement provided E satisfies $E(\mathcal{S} \times^{\mathbf{L}} \mathcal{S}') = E\mathcal{S} \otimes E(\mathcal{S}')$. (E.g. satisfied when $E(\mathcal{S} = \mathcal{S} \otimes^{\mathbf{L}} \mathbb{C}_x$ derived fiber at $x \in X$) In such cases, the eigensheaves are sum of shifted copies of $\mathbb{C}_x$. This all works out well when X is smooth and projective.

On Abelian varieties, dualizing using Fourier–Mukai leads to the notion of Eigensheaves for convolution. Given $m: A \times A \to A$, we put the requirement that $m_*(\mathcal{S}\otimes\mathcal{F}) = Ext(L_\alpha, \mathcal{S})\otimes\mathcal{F}$, for all $\mathcal{S} \in DCoh(A)$. The "fibre functor" $Ext(L_\alpha, \mathcal{S})$ is Fourier–Mukai dual taking fibre at α. Eigensheaves are sums of shifted copies of the L_α's themselves

$$L_\alpha \star L_\alpha = L_\alpha \otimes \Lambda^\bullet(\bar{V}^\vee).$$

In the non-Abelian case, the analogue of A will not be an Abelian group, or even a group. So, we must impose the eigensheaf condition differently.

4.1 Global eigensheaf condition

Given that translations $T_a\colon A \to A$ generate A and that $T_a^* = \mathbb{C}_a\star$ we might try to just impose the eigensheaf condition only on translations, which turns out to be correct, if done in the universal family. Consider $id \times m\colon A \times A \to A \times A$. The global eigensheaf condition on $\mathcal{F} \in DCoh(A)$ is

$$(id \times m)_*(\mathcal{O} \boxtimes \mathcal{F}) = L_\alpha^{-1} \boxtimes \mathcal{F}.$$

Theorem 4.1 *$\mathcal{F}$ is a global L_α eigensheaf if and only if for any $\mathcal{S}$ it is an L_α eigensheaf for $\mathcal{S}\star$.*

Proof: $\mathcal{S}\star\mathcal{F} = m_*(\mathcal{S}\boxtimes\mathcal{F}) = p_*(id\times m)_*(\mathcal{S}\boxtimes\mathcal{F}) = p_*\left[(id \times m)_*(\mathcal{S}\boxtimes\mathcal{O}) \otimes (\mathcal{O}\boxtimes\mathcal{F})\right] = p_*\left[(\mathcal{S}\boxtimes\mathcal{O}) \otimes (L_\alpha^{-1}\boxtimes\mathcal{F})\right] = p_*((\mathcal{S}\otimes L_\alpha^{-1})\boxtimes\mathcal{F}) = Ext(L_\alpha, \mathcal{S}) \otimes \mathcal{F}$. □

It is interesting to investigate the information dual to $BT^\vee$. We have $Dcoh(BT^\vee) = DRep(T^\vee) =$ graded functions on $T^\vee$.

4.2 Deformations of the Fourier–Mukai transform

The honest Fourier–Mukai for $GL(1)$ involves the moduli of flat holomorphic line bundles on Σ. It turns out that Fourier–Mukai can be extended to this setting as well, and leads to non-commutative deformations on the eigensheaf side. Briefly, in the eigensheaf condition we require L to be a flat line bundle.

Just as coherent sheaves are "controlled" by $\mathcal{O}$, flat bundles are controlled by the (sheaf of) algebras of differential operators $\mathcal{D}$. Locally, in coordinates, if $\mathcal{O} = \mathbb{C}[x_1, \cdots, x_n]$, then $\mathcal{D} = \mathbb{C}[x_1, \cdots, x_n, \partial_1, \cdots, \partial_n]$ with $[\partial_i, x_j] = \delta_{ij}$.

$\mathcal{D}$ is filtered by the order of the differential operators: $\mathcal{O} = \mathcal{D}^0 \subset \mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots$ with $\mathcal{D}_k/\mathcal{D}^{k-1} \simeq Sym^k T$; and $gr\mathcal{D} \simeq \oplus_k \mathcal{D}^k/\mathcal{D}^{k-1} \simeq Sym\,T$ as algebras.

$\mathcal{D}$ is a non-commutative deformation of $Sym\,T$, or said in another way, $\mathcal{D}$ is a non-commutative deformation of $\mathcal{O}$ on T^*X, and $\mathcal{D}-$modules are non-commutative deformations of coherent sheaves on $T^*(X)$.

For the non-commutative version of the Fourier–Mukai transform we consider once again: $A = V/L$, $A^\vee = \bar{V}^\vee/L^\vee$ and its deformations. On the $A-$side we consider the non-commutative deformation $\mathcal{D}$ of $A \times V^\vee$ and on the $A^\vee-$side the complex algebraic deformations of $A^\vee \times V^\vee$.

On the $A^\vee-$side we will define a affine fibre bundle $\widetilde{A^\vee} \to A^\vee$ with fibre $V^\vee$. Up to isomorphism such bundles are given by classes in $H^1(A^\vee, V^\vee) \simeq$

$V^\vee \otimes H^1(A^\vee) = V^\vee \otimes \bar{(\bar{V}^\vee)}^\vee = V^\vee \otimes V$; and there is a distinguished class $Id \in V^\vee \otimes V$. This gives the universal extension of $A^\vee$ as an Abelian algebraic group.

Remark 4.2 If you filter the functions on $\widetilde{A^\vee}$ by fibrewise degree, then the corresponding gr is $Sym(V)$. So $\widetilde{A^\vee}$ is a deformation of $Sym(V)$ over A.

When A and $A^\vee$ are isomorphic (e.g. Jacobian, principally polarized case) we get an isomorphism $V \simeq \bar{V}^\vee \simeq T \cdot A^\vee$ from the Chern class of he principal polarization and so we are dealing once again with a deformation of $Sym(T)$.

Theorem 4.3 *$\widetilde{A^\vee}$ is the moduli space of isomorphism classes of flat holomorphic line bundles on A.*

Remark 4.4 All degree zero line bundles on A admit a unique flat unitary connection. But now, we are not requiring unitarity, so we allow the addition of an arbitrary holomorphic $1-$form on A. There is a forgetful morphism from $V^\vee$ to $A^\vee$:

$$\begin{array}{c} V^\vee = \{\text{ global holomorphic differentials on } A\} \\ \Downarrow \\ \text{Mod(flat line bundles on } A) \\ \Downarrow \\ A^\vee. \end{array}$$

This must be classified by a universal element on $H^1(A^\vee, V^\vee) = V \otimes V^\vee$ and the only ones are 0 and scalar multiples of the identity. Now, all scalar multiples of the identity give rise to the same extension, but 0 gives rise to the trivial extension. So, all we are saying is that the extension is non-trivial.

As a complex analytic manifold Mod(flat line bundles on A) is isomorphic to a product of copies of $GL(1)$ labelled by a basis of $\pi_1(A)$. Indeed, flat line bundles on A are classified by elements of $Hom(\pi_1 A, GL(1))$. This shows that $\widetilde{A^\vee} \neq A^\vee \times V^\vee$, because the latter contains a compact submanifold $A^\vee \times 0$ of dimension > 0. However, algebraically this is not true. For example, $H^*(\widetilde{A^\vee}) \simeq H^*(A^\vee)$ remembers the Hodge structure of A and its isomorphism class.

As a consequence, there exists a universal (Poincaré) line bundle on $A \times \widetilde{A^\vee}$ with a flat connection along A varying holomorphically in $\widetilde{A^\vee}$. Ignoring the connection, its the pull-back of the Poincaré line bundle on $A \times A^\vee$. So, we can define a non-commutative version of Fourier–Mukai transform:

$$DCoh(A^\vee) \to D(\mathcal{D} - Mod(A))$$

by pull-back, $\otimes \mathcal{P}$, and push-down by Dolbeault cohomology, and remember the flat connection to get a $\mathcal{D}-$module, not just an $\mathcal{O}-$module.

Note that the result is usually not coherent as an $\mathcal{O}-$module, but it is an a $\mathcal{D}-$module. For example, $\mathcal{O}$ transforms to the $\mathcal{D}-$module generated by the skyscraper sheaf at 1 on A. As an $\mathcal{O}-$module, the result is the Fourier–Mukai transform of $Sym(V)$. As another example, the structure sheaf of the copy of $V^\vee$ over $1 \in A$ (=the variety of opers) maps to $\mathcal{D}$ itself (it is $Sym(V^\vee)$ tensored with the Fourier–Mukai transform of the skyscraper sheaf as an $\mathcal{O}-$module.

Flat vector bundles on A come from $0-$dimensional sheaves on $\widetilde{A^\vee}$. The points of the support indicate the eigenvalues of the connection.

The inverse map requires care. Ordinary Fourier–Mukai would ignore the connection on $\mathcal{P}$ which is clearly wrong. E.g. $\mathcal{O}$ would map to the whole variety of opers (=connections on the trivial bundle). The correct Fourier–Mukai transform form $D(\mathcal{D} - mod(A))$ to $D(Coh\widetilde{A^\vee})$ couples the sheaf to the De Rham complex along A before pushing down.

$$\mathcal{S} \to (\mathbf{R}P_2)_* \left[(p_i^* \mathcal{S}) \otimes \mathcal{P} \otimes (\Omega_A^\bullet, \partial)\right],$$

where $\Omega_A^\bullet$ denotes the holomorphic De Rham complex.

Geometrically this is a deformation of the following parametrized Fourier–Mukai on $\mathcal{O}-$modules over $V^\vee$:

$$\begin{tikzcd}[column sep=small] & A \times A^\vee \times V^\vee \arrow[dl] \arrow[dr] & \\ A \times V^\vee \arrow[dr] & & A^\vee \times V^\vee \arrow[dl] \\ & V^\vee & \end{tikzcd} .$$

Note that in the deformation the map $A^\vee \times V^\vee \to V^\vee$ gets lots (becomes non-holomorphic). Whereas the map $A \times V^\vee \to V^\vee$ deforms to a ring homomorphism $Func(V^\vee) = Sym(V) \to \mathcal{D}(A)$. It is a bit dangerous to think of this as a map, because $Sym(V)$ is not central in $\mathcal{D}(A)$. But we can still use

it. For any $\mathcal{M} \in Coh(V^\vee)$ let $\mathcal{M}' = \mathcal{D}(A) \otimes_{\Gamma(\mathcal{D})} \Gamma(\mathcal{M})$. This lifts coherent sheaves on $V^\vee$ to $\mathcal{D}-$modules.
FACT: the Fourier–Mukai transform of such an $\mathcal{M}$ is supported on the oper copy of $V^\vee$ in $\widetilde{A^\vee}$.

This construction generalizes to the non-Abelian case (Beilinson–Drinfeld construction) and is one of the few geometrically well understood constructions of Hecke eigensheaves in the non-Abelian case. The constructions for the $GL(n)$ case by Laumon–Lafforgue–Gaitsgory requires difficult cohomology calculations.

5 Geometric Langlands V

In this lecture we discuss vector bundles on Riemann surfaces, flat versus Higgs bundles, and the Hitchin system.

NARASIMHAN–SESHADRI THEOREM

Let Σ be a fixed Riemann surface. Recall that by Narasimhan–Seshadri, flat unitary vector bundles of rank n on Σ correspond to representations of $\pi_1(\Sigma)$ into $U(n)$, whereas flat holomorphic vector bundles correspond to representations of $\pi_1(\Sigma)$ into $GL(n)$. We denote

$$M_{U(n)} := Hom(\pi_1\Sigma, U(n))/conjugation\ by\ U(n).$$

The open subvariety $M^o_{U(n)} \subset M_{U(n)}$ corresponding to irreducible representations is also the moduli space of stable algebraic (or holomorphic) vector bundles on Σ of degree zero ($c_1 = 0$). Here E is *stable* iff for any proper holomorphic sub-bundle $F \subset E$ we have

$$\mu(F) = \frac{c_1(F)}{rk\, F} < \frac{c_1(E)}{rk\, E} = \mu(E).$$

The Narasimhan–Seshadri theorem also says that E carries a unique Hermitian–Einstein metric (with curvature a constant multiple of the Kähler form on Σ) and that the holonomy representation into the projective unitary group is irreducible.

COMPLEMENT TO NARASIMHAN–SESHADRI

$M^o_{U(n)}$ is a manifold and inherits an algebraic structure from its interpretation as moduli of algebraic bundles. The real analytic structures match. $M_{U(n)}$ is compact and in fact projective, but singular. For degree c_1 prime to the rank, all holonomy representations are irreducible, and $M_{U(n)}$ is a projective manifold.

Parabolic structures

Another method to make M projective and smooth is to add a parabolic structure at one or more marked points. Chose and element $g \in U(n)$ and a point $x \in \Sigma$ and consider the representation variety:

$$M_{U(n)}(g) = \{f \in Hom(\pi_1(\Sigma - x), U(n)) : f(\alpha) \in g, for\ \alpha\ loop\ at\ x\}/U(n).$$

These are the bundles with parabolic structure of type g at x.

The centralizer $Z(g)$ of g in $U(n)$ is a product of unitary groups (equal to $U(1)^n$ if g has distinct eigenvalues) and the quotient $Fl_g = U(n)/Z(g)$ is the generalized flag variety of $U(n)$. It is a full flag variety if g has n distinct eigenvalues, and a Grassmanian if g has only 2 distinct eigenvalues.

Theorem 5.1 *(Mehta-Seshadri–Ramanathan) The subset of irreducible elements in $M_{U(n)}(g)$ is the space of isomorphism classes of stable parabolic vector bundles with a flag of type Fl_g in the fibre over $x \in \Sigma$.*

The variety Fl_g can be identified with the set of all flags $0 = F_0 \subset F_1 \subset \cdots \subset F_k = \mathbb{C}^n$, where k is the number of distinct eigenvalues of g, and $\dim F_p/F_{p-1}$ is the multiplicity of the $p-$th eigenvalue of g. Morally, we have a fibre bundle

$$\begin{array}{ccc} Fl_g & \hookrightarrow & M_{U(n)}(g) \\ & & \downarrow \\ & & M_{U(n)} \end{array} \qquad (*)$$

but this is only strictly true on the complement of a closed subvariety of the moduli space; it breaks whenever the vector bundle has holomorphic automorphisms that are broken by the choice of flag. It also breaks because the stability condition on the parabolic bundle is not the same as on the underlying bundle.

Still the fibration $(*)$ holds on the ideal world of algebraic stacks which remember the automorphisms of the bundle. This is important, because the Hecke eigensheaf condition is defined in terms of these parabolic bundles.

5.1 Hecke correspondence for $GL(n)$

For $1 \leq k \leq n$, let $\mathfrak{M}_d^k$ denote the moduli (stack) of algebraic vector bundles of degree d on Σ equipped with a $k-$dimensional subspace in the fibre over x. We can map $\mathfrak{M}_d^k$ to $\mathcal{M}^0$ in 2 different ways:

i) Forget the subspace in the fiber over x.

ii) Consider the sheaf of sections with simple poles at x, whose principal parts take values in the distinguished subspace. This results in a bundle of degree $d+k$.

The diagrams

$$\begin{tikzcd} & \mathcal{M}_d^k \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M}_d^0 & & \mathcal{M}_{d+k}^0 \end{tikzcd}$$

are the elementary Hecke transformations.

Note: There are Hecke correspondences associated to every co-weight of $GL(n)$. The elementary ones are associated to the fundamental co-weights $(1, 1, \cdots, 1, 0, \cdots, 0)$ and under Langlands duality to the exterior power maps $\Lambda^k \mathbb{C}^n$ of the dual group (which happens to be $GL(n)$ in this case). For $GL(n)$ we will be able to state the Hecke eigensheaf condition just in terms of the elementary Hecke correspondences (which generate the ring of Hecke correspondences). For other groups we need a generalization of the construction which uses Schubert varieties of the Loop Grassmanian.

If you let x vary in Σ there is a universal Hecke correspondence. Set $\mathcal{M}(k, \Sigma)$ to be the moduli of bundles with a flag at a point varying in Σ and write:

$$\begin{tikzcd} & \mathcal{M}(k, \Sigma) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M}_d^0 \times \Sigma & & \mathcal{M}_{d+k}^0 \times \Sigma \end{tikzcd}$$

and the Hecke $V-$ eigensheaf condition on $\mathcal{F}$ is

$$\mathbf{R}q_* p^* \mathcal{F} = \mathcal{F} \boxtimes V,$$

where V is a vector bundle on Σ in the category of flat line bundles ($\mathcal{D}-$modules), and $\mathbf{R}q_*$ is the de Rham cohomology along the fibre.

For $GL(1)$ the Hecke map $\mathcal{H}_1 \colon J_q \to J_{d+1}$ is an isomorphism.

5.2 The stack of bundles

Even for $GL(1)$ to get Fourier–Mukai to work properly, we had to take into account the automorphisms of line bundles. That was quite easy, because all line bundles have the same automorphism group, the centraliser of the holonomy of the connection. It is even more important to do that now, when the isomorphism groups vary with the bundle.

E.g. If V is a stable bundle, then V has only the scalar automorphisms. For polystable bundles, the holomorphic automorphism group is the complexification of the unitary automorphism group. For $V = L \oplus L^{-1}$, the tangent space to the automorphism group is $\Gamma(\Sigma, End(V)) = \Gamma(\Sigma, \mathcal{O} \oplus \mathcal{O} \oplus L^2 \oplus L^{-2})$ which gets large with the degree of L. Note that it is the off diagonal part that leads to this new behaviour.

Stacks were invented to describe moduli of objects with automorphisms. The stack of vector bundles is any category (groupoid) in which:

X_0= the objects, parametrize *all* vector bundles on Σ,

X_1= the morphisms, parametrize *all* isomorphisms,

and such that X_0, X_1 and the composition have been given algebraic structures.

Such models exist by a general "Quot-scheme" construction of Grothendieck, refined by Gieseker. But, in this case it is easier to give concrete (analytic) models.

5.3 Segal's double coset construction

Let Δ be a disc on Σ with boundary circle $\partial\Delta$ and denote $\Sigma^\circ = \Sigma - \Delta$,

$LGL(n)$ = smooth maps $\partial\Delta \to GL(n)$

$GL(n)[\Delta]$ = boundary values of holomorphic maps $\Delta \to GL(n)$ with C^∞ boundary values

$GL(n)[\Sigma^\circ]$ = boundary values of holomorphic maps $\Sigma^\circ \to GL(n)$ with C^∞ boundary values, and

$$X_{\Sigma^\circ} := LGL(n)/GL(n)[\Sigma^\circ].$$

$GL(n)[\Delta]$ acts on the left on X_{Σ° and the stack of vector bundles on Σ is the variety X_{Σ° with the action of $GL(n)[\Delta]$.

E.g. Coherent sheaves on $\mathcal{M}$ are $G(\Delta)$- equivariant coherent sheaves on X_{Σ°. The infinite dimensionality is misleading, because the action of the group has finite dimensional slices everywhere.

By a theorem of Grauert, every holomorphic vector bundle on Σ becomes trivial upon restriction to Σ° and to Δ. Grauert says that holomorphic vector bundles on a Stein manifold are classified by their topological type. So, the only information is in the gluing map, a smooth map $\partial\Delta \to GL(n)$. But changing the transition function by an automorphism on the right or left leads to isomorphic $GL(n)$ bundles. So,

$$\mathfrak{M} = GL(n)\backslash LGL(n)[\Delta]/GL(n)[\Sigma^\circ]$$

is the set of isomorphism classes of vector bundles. Moreover, the stabilizer of the $GL(n)[\Delta] \times GL(n)[\Sigma^\circ]$ action on $GL(n)$ at some loop λ is precisely the group of automorphisms of the bundle, because the two automorphisms of either side assemble to give an automorphism of the bundle.

To the action of a group G on a space X we associate the "action groupoid" with space of objects X, space of morphisms $G \times X$, source maps the projection to X, and target map the action $(g, x) \mapsto gx$. In our case, the action groupoid for $GL(n)[\Delta]$ on X_{Σ° is equivalent to the stack of vector bundles on Σ.

Remark 5.2 (ι) the notion of equivalence of stacks is delicate to define. It requires some technology of Grothendieck topologies, sheaf theory,etc.
($\iota\iota$) The above construction can be made algebraic, turning a variant of X_{Σ° into a scheme and $GL(n)[\Delta]$ into an algebraic group.
($\iota\iota\iota$) There is a much better known model for $\mathcal{M}$, the Atiyah–Bott quotient construction, which is better for many purposes, but can not be made algebraic.

5.4 Higgs fields and the cotangent stack of $\mathcal{M}$

Unlike the Abelian case, the Langlands correspondence for arbitrary G does not seem to have a formulation directly in terms of $\mathcal{M}$. Instead, what is involved is the total space of the cotangent bundle of M and its deformations:

non-Abelian to the differential operators on $\mathcal{M}$, and Abelian to non-unitary local systems on Σ.

The cotangent bundles has a natural partial compactification to the moduli of Higgs bundles. There is a corresponding moduli stack of Higgs bundles, which is just $T^*\mathcal{M}$. This is a beautiful example of a completely integrable algebraic system, with a good reason for being so. It has been generalizes by Simpson and Corlette to arbitrary Kähler manifolds in place of Σ, but many of the beautiful features are present only for curves.

Definition 5.3 A *Higgs field* for a vector bundle $E \to X$ is a holomorphic map $\theta\colon E \to E \otimes \Omega^1_{hol}$ with the property that $\theta \wedge \theta\colon E \to E \otimes \Omega^2_{hol}$ is zero.

Obviously, for Riemann surfaces, the second condition is vacuous. This condition comes from the fact that Ω is a degenerate flat holomorphic connection on E over X.

$$F = (\partial + \theta)^2 = \partial^2 + [\partial, \theta] + \theta \wedge \theta$$

If we scale θ by t, giving $\partial + t\theta$, and let $t \to \infty$ we obtain the condition $\theta \wedge \theta = 0$. The remarkable theorem is that (with a stability condition) this degeneration can be reversed and a flat (non-unitary) connection can be constructed canonically from θ and the Kähler metric on X.

Definition 5.4 A Higgs bundle (E, θ) is *stable* (resp. *semistable*) if every $\theta-$invariant subbundle has strictly lower slope (resp. $\leq$) than E. A Higgs bundle is *polystable* if it is the direct sum of stable Higgs bundles.

Given a metric on a Higgs bundle, define a connection as follows: $D^{1,0} = \partial + \theta,\ D^{0,1} = \bar{\partial} + \bar{\theta}$, where $\bar{\partial} = \theta^* d\bar{z}$. Note that although $\partial + \bar{\partial}$ is Hermitian, $D = D^{1,0} + D^{0,1}$ is not Hermitian, because of the wrong sign in θ. The curvature of D is $F = [\partial, \bar{\partial}] + [\theta, \bar{\theta}]$.

Definition 5.5 D is *Hermitian–Yang–Mills* if $\omega \lrcorner F = \lambda \cdot Id$.

λ is then the first Chern slope $c_1(E)/(rk\, E)$ and D defines a structure on E that is as flat as possible, subject to the topological constraint.

Conversely, given a flat vector bundle E on X with connection D and a metric not necessarily preserved by the connection, define $\theta: E \to E \otimes \Omega^1$ such that the $(1,0)$ and $(0,1)$ parts of D satisfy:

$$D^{1,0} = \partial + \theta, \ D^{0.1} = \bar{\partial} + \bar{\theta}$$

and $\partial + \bar{\partial}$ is a Hermitian connection for the given metric. Indeed, θ is uniquely determined by the condition: $(\partial - \theta) + D^{0,1}$ is the unique Hermitian connection compatible with $D^{0,1}$ and then $\theta = (D^{1,0} - (\partial - \theta))/2$. Note that θ need not be a Higgs field: $\bar{\theta} \neq 0$; moreover, if dim $X > 1$, we might have $\bar{\theta}^2 \neq 0$, and $\theta \wedge \theta \neq 0$.

The metric is *harmonic* (with respect to D) if $\bar{\theta} = \bar{\partial}^2 = \theta \wedge \theta = 0$. In this case, then $\bar{\partial}, \theta$ define a Higgs bundle. Note that a metric on E is the same as a reduction of the GL frame bundle of E to a unitary frame bundle, that is, a section of the contractible bundle with fibre $GL(n)/U(n)$. This bundle has a flat structure inherited from E.

Proposition 5.6 *(Siu, Corlette, et al) The metric is harmonic iff the associated section of the $GL(n)/U(n)$ bundle is harmonic in the differential geometric sense, i.e. the Laplacian vanishing.*

Theorem 5.7 *(*Higgs flat equivalence - *Hitchin, Siu, Corlette, Donaldson, Simpson)*

1. A flat bundle has a harmonic metric iff the associated monodromy representation $\pi_1 X \to GL(n)$ is completely reducible.

2. A Higgs bundle admits a Hermitian–Yang–Mills metric iff it is polystable as a Higgs bundle. If $c_1 = c_2 = 0$ this metric is harmonic and the connection is flat.

3. The moduli spaces of semisimple flat bundles and polystable Higgs bundles are real-analytically isomorphic.

5.5 The Hitchin system

Let $\mathcal{M}$ be the moduli space of stable vector bundles of fixed rank and Chern class on the Riemann surface Σ. Hitchin studied the differential and symplectic geometry of $T^*\mathcal{M}$. Note that

$$T^*_E\mathcal{M} = H^1(\Sigma, End(E))^\vee \simeq H^0(\Sigma, End(E)^\vee \otimes L).$$

So, a cotangent vector on $\mathcal{M}$ is a Higgs field on Σ!

There are some natural global holomorphic functions on $T^*\mathcal{M}$: any invariant polynomial on gl_n of degree d gives a map from $T^*\mathcal{M}$ to $H^0(\Sigma, K^{\otimes d})$ by pointwise application to the Higgs field. The basic invariant polynomials (the coefficients of the characteristic polynomial $\det(tI - A)$) give a holomorphic map:

$$\chi\colon T^*\mathcal{M} \to \bigoplus_{d=1}^{n} H^0(\Sigma, K^{\otimes d}).$$

The target $\mathcal{H} := \bigoplus_{d=1}^{n} H^0(\Sigma, K^{\otimes d})$ is called the *Hitchin space.*

Note: $\dim \mathcal{H} = 1 + \sum_{d=1}^{n}(2d-1)(g-1) = 1 + (g-1)n^2 = \dim M$. Recall that on $T^*\mathcal{M}$ we have a holomorphic symplectic form $\omega = dp \wedge dq = d(pdq)$ in local coordinates $(q_a, p^a = d\xi^a)$.

Theorem 5.8 *(Hitchin) Assume $g \geq 2$.*

1. *Global holomorphic functions on $T^*\mathcal{M}$ are pulled back from $\mathcal{H}$ via χ. Hence, $\Gamma(T^*\mathcal{M}, \mathcal{O}) = \mathcal{O}(\mathcal{H})$.*
2. *χ is flat. In particular, all fibres have dimension* $\dim M$.
3. *The fibres of χ are Lagrangian holomorphic subvarieties. Equivalently, $\omega|_{fibre} = 0$ and the global holomorphic functions Poisson-commute.*
4. *The generic fibres are Zariski open subsets of Abelian varieties. The global holomorphic Hamiltonians define linear flows along those varieties.*

Some ideas of the proof given by Beauville–Narasimhan: the symplectic form ω is exact on $T^*\mathcal{M}$ consequently, it is exact on $\mathcal{M}_{Higgs}$, but the fibres are compact Kähler, so exact holomorphic differentials are null. Hence, the fibres are Lagrangian.

In the case $g = d = 2$ or for singular spaces, then the argument of Beauville–Narasimhan fails. However, there is a more general argument based on hyperkähler reduction that goes through then.

Note: an algebraic version of 1. also holds with $Sym\mathcal{H}^*$ instead of $\mathcal{O}(\mathcal{H})$, that is, with polynomials in place of holomorphic maps.

6 Geometric Langlands VI

6.1 Relative compactification

Theorem 6.1 *The moduli space $\mathcal{M}_{Higgs}$ of semi-stable Higgs bundles contains $T^*\mathcal{M}$, and the complement has dimension $\geq (g-1)(n-1)$. In particular, χ extends to an algebraic map. In the case when the degree and the rank are co-prime, then $\mathcal{M}_{Higgs}$ is smooth.*

Theorem 6.2 *$\mathcal{M}_{Higgs}$ is proper over $\mathcal{H}$, hence $\mathcal{M}_{Higgs}$ can be regarded as a relative compactification of the Hitchin space $\mathcal{H}$.*

Corollary 6.3 *All holomorphic functions on $T^*\mathcal{M}$ are lifted from $\mathcal{H}$.*

6.2 Holomorphic symplectic construction of $\mathcal{M}_{Higgs}$

We recall the Atiyah–Bott construction of the moduli space of connections on the topologically trivial bundle. Let $\mathfrak{g}_{\mathbb{C}} = gl(n, \mathcal{C})$ and let $\mathcal{A} = \{(0,1)\text{-forms}$ with values in $\mathfrak{g}_{\mathbb{C}}\}$. Let $\mathcal{G}_{\mathbb{C}}$ denote the complex gauge group (automorphisms of the bundle covering the identity), and $L(\mathcal{G}_{\mathbb{C}})$ its Lie algebra. An element $\xi \in L(\mathcal{G}_{\mathbb{C}})$ acts by

$$\alpha \mapsto -(\bar{\partial}_\alpha \xi + [\alpha, \xi]) = [ad_\xi, \bar{\partial}_\alpha].$$

Inside $\mathcal{A}$ there is the open dense subset $\mathcal{A}^s$ parametrizing stable bundles.

Theorem 6.4 *$\mathcal{A}^s/\mathcal{G}_{\mathbb{C}}$ is isomorphic to the moduli space of stable vector bundles.*

Note that in the above construction vector bundles are topologically trivial. In general for other Chern classes, one must replace $\mathcal{G}_{\mathbb{C}}$ and $\mathcal{A}$ with connections and gauge transformations on a principal bundle of a chosen topological type.

Now let $T^*\mathcal{A}$ be the total space of the cotangent bundle

$$T^*\mathcal{A} \simeq \Omega^{0,1}(\Sigma, \mathfrak{g}_{\mathbb{C}}) \times \Omega^{1,0}(\Sigma, \mathfrak{g}_{\mathbb{C}})$$

which has the holomorphic symplectic form

$$(\alpha, \varphi)(\alpha', \varphi') \mapsto \int_\Sigma Tr(\alpha\varphi' - \alpha'\varphi)$$

and the holomorphic moment map $\mu\colon \mathcal{A} \to L(\mathcal{G}_{\mathbb{C}}^*) = \Omega^{1,1}(\Sigma, \mathfrak{g}_{\mathbb{C}})$ given by

$$(\alpha, \varphi) \mapsto \bar{\partial}\varphi + [\alpha, \varphi],$$

which is simply the dual to the derivative of the action of $L(\mathcal{G}_{\mathbb{C}})$ on $Vect(\mathcal{A})$. Then

$$\mu^{-1}(0) = \{\varphi : \bar{\partial}_\alpha \varphi = 0\}$$

is the set of holomorphic Higgs fields, and

$$\mathcal{M}_{Higgs} \simeq \mu^{-1}(0)^s/\mathcal{G}_{\mathbb{C}} \supset T^*(\mathcal{A}^s/\mathcal{G}_{\mathbb{C}}).$$

Corollary 6.5 *Functions of φ alone Poisson commute.*

6.3 Spectral curve and meaning of the fibres

Consider a Higgs field θ on a bundle E over Σ. Given (E, θ) we define a coherent sheaf $\mathcal{F}_{E,\theta}$ on $T^*\Sigma$ (= total space of K) whose direct image to Σ is E. The support of $\mathcal{F}_{E,\theta}$ depends on $\chi(\theta)$ alone and plays the role of the "set of eigenvalues" of θ. The sheaf $\mathcal{F}_{E,\theta}$ itself is defined via the eigenspace decomposition of E and the spectrum of θ. Generically, where the spectral curve has no simple singularities, this can be made precise with the naive meaning of eigenvalue, but a sheaf theoretic formulation works in all cases.

Analogy: if E is a vector space, $\theta \in End(E)$ and P_θ the characteristic polynomial of θ, then by Hamilton–Cayley, $P_\theta(\theta) = 0$. So, E becomes a module over $\mathbb{C}[t]$, supported on the subscheme $S = Spec(\mathbb{C}[t]/P_\theta(t))$, which is 0-dimensional of degree n. This is the "spectral scheme" of θ and E becomes a coherent sheaf over it (= module over $\mathbb{C}[t]/P_\theta(t)$).

If θ has distinct eigenvalues, then S is reduced, has n distinct points, and E is a line bundle over S. When θ has repeated eigenvalues, S has nilpotents and the sheaf $\mathcal{F}_{E,\theta}$ may not be a line bundle over S.

Example 6.6 (i) If $\theta = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, then $S = \mathbb{C}[t]/(t-a)^2$. In this case $\mathcal{F}_{E,\theta}$ has torsion killed by $(t-a)$ and the fibre at a has rank 2.

(ii) If $\theta = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, then $S = \mathbb{C}[t]/(t-a)^2$. In this case $\mathcal{F}_{E,\theta}$ is a line bundle and the fibre at a is the "eigen-quotient" (not the eigen-vector).

Lemma 6.7 *For $\theta \in gl(n)$, the following are equivalent:*

1. *All Jordan blocks of θ have maximum size.*

2. *The centralizer of θ is Abelian.*

3. *The centralizer of θ has dimension n.*

4. *θ is conjugate to*

$$\begin{bmatrix} p_1 & p_2 & \cdots & & p_n \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & 1 & 0 \end{bmatrix} \tag{1}$$

Definition 6.8 θ is called regular if it satisfies the properties above.

In general, $\mathcal{F}_{E,\theta}$ is a line bundle on $Spec(\theta)$ iff θ is everywhere a regular matrix.

Proposition 6.9 *Matrices of the form (1) give a global section of the adjoint action of $GL(n)$ on $gl(n)$.*

The projection $gl(n) \to gl(n)//GL(n)$ has a section called the *Kostant section.* More generally, such a section exists for every reductive group G. The principal $sl_2 \subset \mathfrak{g}$ decomposes $\mathfrak{g}$ into $\oplus \mathbb{C}^{2m_i+1}$ where the m_i's are the exponents of $\mathfrak{g}$; they are $0, \cdots, n-1$ for $gl(n)$. The Kostant slice picks the space generated by the highest weight vectors in each component. Hitchin's spectral cover construction is a global version of this construction.

Given E, θ over Σ at each point $x \in \Sigma$, $\theta_x\colon E_x \to E_x \otimes K_x$ makes E_x into an algebra over $\mathbb{C}[T_x]$. Varying x over Σ defines a $Sym(T)$-module. The *spectral curve* for the Hitchin map $T^*M \to \bigoplus_i \Gamma(\Sigma, K^{\otimes m_i+1})$ is the curve on $Tot(K)$ defined by the equation

$$\theta^n + p_1\theta^{n-1} + \cdots + p_n = 0.$$

The fibre of $\mathcal{F}_{E,\theta}$ over some $(x,t) \in Tot(K)$ is just $\pi^*(E)/(M_x, t-\theta)$.

Theorem 6.10 *For a generic value of $\chi(\theta)$ the spectral curve is smooth, irreducible, with simple branch points over Σ.*

7 Geometric Langlands VII

Recall the Hitchin map:

$$\begin{array}{rcl} \chi\colon T^*M & \to & \bigoplus_{d=1}^n H^0(\Sigma, K^{\otimes d}) \\ \theta & \mapsto & \text{coeffs. char. poly. of } \theta \end{array}.$$

A fixed value of χ determines a spectral curve S in $Tot(K_\Sigma)$ whose fiber over each $x \in \Sigma$ is the spectral scheme of θ in the line K_x.

The projection $\pi\colon S \to \Sigma$ is finite of degree n. The action of $\theta\colon E \to E \otimes K$ turns E into a sheaf $\mathcal{F}_{E,\theta}$ over $Tot(K_\Sigma)$, supported on S; we have $\pi_*\mathcal{F}_{E,\theta} = E$ over Σ. The singularities of $\pi\colon S \to \Sigma$ occur where eigenvalues coincide, that is, on the zeroes of the discriminant of the characteristic polynomial of θ, $\chi_\theta = t^n + p_1 t^{n-1} + \cdots + p_n$.

The discriminant is a section of $K^{n(n-1)}$ so the curve S has $n(n-1)$ branch points. For generic values of χ (away from the "discriminant locus" in $\mathcal{H}$) these are simple zeroes, and S is an $n-$fold branched cover with simple branching over $(2g-2)(n-1)n$ points of Σ. So, in this case, S is smooth of genus g' given by the Riemann–Hurwitz formula

$$2 - 2g' = n(2-2g) + (2g-2)(n-1)n,$$

hence

$$g' = n^2(g-1) + 1.$$

Example 7.1 For $GL(2)$ the genus of S is $4g-3$.

When S is smooth, $\mathcal{F}_{E,\theta}$ must be a line bundle over S : it can have no torsion because $\pi_*\mathcal{F}_{E,\theta}$ has none, and must have rank 1 because $\pi_*\mathcal{F}_{E,\theta}$ has rank n. Conversely, for each line bundle L over the spectral curve, π_*L is a rank n vector bundle over Σ, with Higgs field.

Proposition 7.2 *When S is smooth π_*L is stable for generic L.*

Remark 7.3 π_*L is always stable as a Higgs bundle, because a Higgs subbundle of π_*L would have to be a subsheaf of L on the spectral curve.

We get a correspondence:

$$\{\text{Higgs bundle with prescribed } \chi\text{-image}\}$$
$$\Updownarrow$$
$$\{\text{line bundle on the spectral curve}\}$$

which is bijective, except for the stability issue mentioned above. At the stack level — thus, without considering stability — this correspondence is truly bijective, and we have

$$\{\text{Higgs bundle on } \Sigma\}$$
$$\Updownarrow$$
$$\{\text{sheaf on } Tot(K_\Sigma)\text{, which is torsion free of rank } n \text{ over } \Sigma\ \}$$

Remark 7.4 $\deg \pi_* L = \deg L - n(n-1)(g-1)$, the defect arising because of the branch points.

Remark 7.5 For $SL(n)$ instead of $GL(n)$ the condition $\det \pi_* L = 1$ imposes a condition on the Jacobian of the spectral curve that cuts out a linear subvariety of codimension g. Because of the "defect" in the direct image, this is *not* an Abelian subvariety.

Example 7.6 For $SL(2)$ the spectral curve is a double cover branched at $4g-4$ points. Pairing them leads to a picture:

PICTURE GOES HERE

In this case, the covering $S \to \Sigma$ is actually Galois, with deck transformations the sign involution on K_Σ : this is because $Tr(\theta) = 0$ and so the characteristic polynomial has the form $t^2 - \det \theta$. Topologically, the involution switches the two copies of the base curve in the picture above. With respect to this involution, the homology of the spectral curve decomposes as

$$H_1(S) \simeq H_1(\Sigma) \otimes R \oplus (\mathbb{Z}^{4g-4})_-$$

where R is the regular representation of the Galois group $\mathbb{Z}/2$, and the $-$ refers to the sign representation. So, topologically (but not holomorphically)

$$J(S) \simeq J(\Sigma) \otimes R \times J'_-,$$

for a torus J'_- with the sign action of $\mathbb{Z}/2$.

If we have a line bundle L with $\det \pi_* L$ trivial, then all other line bundles with this property have the form $L' = L \otimes L''$ with $\sigma^* L'' = (L'')^{-1}$ in the degree zero Jacobian of S. This picks up an affine copy of the anti-diagonal in $J(\Sigma) \otimes R$ and the entire component J'_-. Now, the principal polarisation on $J(S)$ is given by $\det^{-1}$ of the index bundle, $\det H^1(S, L)/\det H^0(S, L)$. Note that, since $H^*(S, L) \simeq H^*(\Sigma, \pi_* L)$, and $\det^{-1} H^*(\Sigma, \pi_* L)$ is the Narasimhan–Seshadri generator of $Pic(M_{GL(n)})$, we see that the latter pulls back to become the principal polarization in the Jacobian of the spectral curve. But, observe that the principal polarization on $J(S)$ does *not* restrict to a principal polarization on the anti-diagonal in $J(\Sigma)$. The moral is:

- The $SL(2)$ Hitchin fibre is a *Prym variety* inside $J(S)$;
- The principal polarization restricts to a line bundle of degree 2^g.

Recall that a Prym variety is a fixed variety of a finite group action on an Abelian variety. Here, the action is $L'' \mapsto (\sigma^* L'') \times$ (a translation).

A priori, these Prym varieties are not Abelian (they have no distinguished choice of identity); but these ones do have a base point once a $\sqrt{K}$ has been chosen. We'll see a distinguished section of the Hitchin morphism associated to any square root of K on Σ.

7.1 The zero fibre of the Hitchin map

We now discuss the zero fibre $\chi^{-1}(0)$ of the Hitchin morphism. Note that the Higgs field θ is nilpotent if and only if its characteristic polynomial is identically zero. In that case, the spectral scheme S_0 is the $n-$th neighborhood of the zero section. Choosing the trivial line bundle over S_0 leads to

$$\pi_* \mathcal{O} \simeq \mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{-n+1}$$

on Σ, and the Higgs field

$$\theta_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

gives the shift $K^{-i} \xrightarrow{\sim} K^{-i-1} \otimes K$.

Remark 7.7 This $\pi_*\mathcal{O}$ is stable as a Higgs bundle, but strictly unstable as a bundle.

There are many possible types of sheaves $\mathcal{F}_{E,\theta}$ over S_0 which are torsion free over Σ. For definiteness, let us take $GL(2)$. Then we have the following discrete classification:

1. $\mathcal{F}_{E,\theta}$ restricted to the zero section has rank 2. In this case it is isomorphic to E, $\theta \equiv 0$, and we get the zero section of the cotangent bundle $T^*\mathcal{M}$. Note that, when the Higgs field vanishes, stability of E as a bundle and a Higgs bundle agree.

2. $\mathcal{F}_{E,\theta}$ restricted to the zero section has rank 1, then it is a line bundle, say L. It is torsion free, because $\pi_*\mathcal{F}_{E,\theta} = E$ is so. In this case $L = E/Im(\theta\colon E \otimes K^{-1} \to E)$, and we have an extension
$$0 \to L' \to E \to L \to 0$$
with $\theta\colon L \to L' \otimes K$ the Higgs field.

Case 2 has several sub-cases. To simplify slightly, assume now that the group is $SL(2)$; then $L' \simeq L^{-1}$ and the existence of a non-zero θ implies $2\deg L \leq \deg K$ hence $\deg L \leq g-1$ and we have the following possibilities:

- If $\deg L < 0$ the Higgs bundle is unstable (L' is a Higgs sub-bundle of positive slope). Thus, over the unstable part of $\mathfrak{M}$ the zero fibre of the Hitchin morphism has lots of components.

- If $\deg L \geq 0$, E could be (semi)-stable, and indeed those components of the (stack of) Higgs bundles do appear in the $0-$fibre of the stable Higgs moduli space.

- Extreme case: if $\deg L = g-1$, then $\theta\colon L \to L^{-1} \otimes K$ is an isomorphism and $L = K^{1/2}$ is a square root of K; in such case E is an extension of the form
$$K^{-1/2} \to E \to K^{1/2}$$
with $\theta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This component is in the closure of the vector space
$$Ext^1(K^{1/2}, K^{-1/2}) = H^1(K^{-1}) = H^0(K^{\otimes 2})^\vee$$

dual to the Hitchin space, and indeed the linear Hitchin Hamiltonians give the linear flow on this vector space. The tangent to this component of the $0-$fibre is dual to the cotangent space of the case via the symplectic form. In this sense, this vector space is the "limit" of the Abelian varieties as they approach the $0-$fibre.

At the "boundary" of these vector spaces (rescaling the extension class in such a way that the Higgs field $\to 0$) we obtain a divisor in the moduli space, namely, that of bundles which are extensions of $K^{1/2}$ by $K^{-1/2}$. There is one such divisor for each $\sqrt{K}$ in the $SL(2)$ case, but all these divisors coincide for $PSL(2)$. Also, there is only one component for $GL(2)$, but it carries and extra $Jac(\Sigma)$ factor from the determinant; the components for $SL(2)$ correspond to 2-torsion points in this Jacobian.

7.2 Geometric meaning of the zero fibre

There is a distinguished Morse function on the moduli space of stable Higgs bundles, the square norm of the Higgs field, whose Hamiltonian for the Kähler symplectic form is the circle action on $T^*\mathcal{M}$. The $0-$fibre of χ is the union of the *unstable Morse strata*, the unions of downward gradient flows from the critical points. The top critical points are the bundles $K^{-1/2} \oplus K^{1/2}$ with $\theta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the bottom critical set is the zero section of $T^*\mathcal{M}$. In particular, $\mathcal{M}_{Higgs,stable}$ is homotopy equivalent to $\chi^{-1}(0)$. The Hodge structure on its cohomology is pure. A topological consequence of this fact is that the circle-equivariant cohomology of $\mathcal{M}_{Higgs}$ factor as $H^*(\mathcal{M}_{Higgs}) \otimes H^*(BS^1)$; this helps in computations.

7.3 The Hitchin section

Let $E = \mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{-n+1}$ for $GL(n)$. (For $SL(n)$ we would need $E = K^{(n-1)/2} \oplus \cdots \oplus K^{(-n+1)/2}$ and this requires a choice of $\sqrt{K}$ for n even.) Over each point $(p_1, \cdots, p_n) \in \bigoplus_{d=1}^n \Gamma(\Sigma, K^{\otimes d})$ of the Hitchin space, consider the Higgs field

$$\theta_p = \begin{bmatrix} p_1 & p_2 & p_3 & \cdots & p_n \\ -1 & 0 & & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 0 \end{bmatrix} \tag{2}$$

on E. This is a regular Higgs field and defines a regular section of the Hitchin map. Let now $H = (H_0, \cdots, H_{n-1}) \in \bigoplus_{d=1}^{n} H^1(\Sigma, T^{\otimes d-1})$ be a linear Hamiltonian on $T^*\mathcal{M}$

Proposition 7.8 *Under the Hamiltonian flow* $\exp(H)$ *the pair* (E_0, θ) *gets sent to* (E_H, θ) *where* $E_H \simeq_{top} E_0$ *topologically, but with complex structure*

$$\bar{\partial}_H = \bar{\partial}_0 + H_0 + H_1\theta + \cdots + H_{n-1}\theta^{n-1}.$$

Note that here $H_i \in H^1(\Sigma, T^{\otimes i})$ and $\theta^i \in H^0(\Sigma, K^{\otimes i} \otimes End(E))$, hence the products are in $H^1(\Sigma, E)$ giving a variation of the $\bar{\partial}$ operator. For the proof, compare with the expression of the symplectic form in the Atiyah–Bott realization.

Proposition 7.9 *The sweep of the Hitchin section consists of all (pointwise) regular Higgs fields in the respective component.*

Proof. A regular Higgs field determines a topological decomposition of the bundle as $\bigoplus_{j=0}^{n-1} K^{-j}$. However, the only restriction on the $\bar{\partial}$ operator is that it should commute with θ. Now, by regularity of θ, the centraliser of θ in $End(E) = span < 1, \theta K^{-1}, \cdots, \theta^{-n+1}K^{\otimes n} >$. So, in fact,

$$\bar{\partial}_H = \bar{\partial}_0 + H_0 + H_1\theta + \cdots + H_{n-1}\theta^{n-1}$$

with $H_i \in H^1(\Sigma, T^{\otimes i})$ as above.

8 Geometric Langlands VIII

8.1 The partial Poincaré bundle

Motivation: Let A be an Abelian variety with principal polarisation defined by a line bundle $L \to A$. This principal polarisation defines an isomorphism

$A \to A^\vee$, and consequently we have the Poincaré line bundle $\mathcal{P}$ defined over $A \times A^\vee$. Let $m: A \times A \to A$ be the multiplication and $p_1, p_2: A \times A \to A$ be the projections.

Proposition 8.1 $\mathcal{P} = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$.

For a proof see e.g. Polishchuck.

Now, in the case of the Jacobian, the principal polarisation is defined by $\det^{-1}$ of the index bundle, where det is the determinant of the cohomologies $\det = \det H^1(\Sigma, \mathcal{L}) \otimes \det H^0(\Sigma, \mathcal{L})$ of the universal line bundle on $\Sigma \times J$. In the case $A = J =$ the moduli space of line bundles, and m corresponding to $\otimes$, we can write

$$\mathcal{P} = \det^{-1}(\mathcal{L}_{a_1} \otimes \mathcal{L}_{a_2}) \otimes \det(\mathcal{L}_{a_1}) \otimes \det(\mathcal{L}_{a_2}),$$

for $a_1, a_1 \in J$. Now on $T^*\mathcal{M}$ we have the principal polarisation given by det^{-1} defined everywhere. Moreover, on $(T^*\mathcal{M})^{regular}$ the subvariety of regular Higgs fields has an Abelian group structure relative to the Hitchin map χ, with the operation of tensoring 2 line bundles on the spectral curve. Note that $T^*\mathcal{M}^{reg} \times_{\mathcal{H}} T^*\mathcal{M}^{reg}$ consists of pairs of line bundles on the spectral curve. We can in fact extend this operation to a partial addition

$$R := T^*\mathcal{M} \times_{\mathcal{H}} T^*\mathcal{M} - T^*\mathcal{M}^{sing} \times_{\mathcal{H}} T^*\mathcal{M}^{sing} \to T^*\mathcal{M}$$

by tensor product on the spectral curve. This continues to be well-behaved, provided one of the sheaves $\mathcal{F}_{E,\theta}$ is a line bundle. (However, this operation becomes discontinuous when both factors wander off into $T^*\mathcal{M}^{sing}$; e.g. for 2 rank 2 vector bundles on $\Sigma \subset Tot(K_\Sigma)$, their tensor product has rank 4. So, this rules gives no addition law $m \times m \to m$ on the moduli space of vector bundles.)

So, we can define $\mathcal{P}$ over a dense open subset of $T^*\mathcal{M} \times_{\mathcal{H}} T^*\mathcal{M}$. Note that $\mathcal{P}$ is completely defined along the fibres of the projection to $T^*\mathcal{M}^{reg}$. Consequently, we know the Fourier transform of any $\mathcal{F} \in Coh(T^*\mathcal{M})$ when restricted to $T^*\mathcal{M}^{reg}$, and the complete Fourier transform of sheaves supported in $T^*\mathcal{M}^{reg}$ (e.g.points).

Nonetheless, we can not check too much about $\mathcal{P} \circ \mathcal{P}$ because part of the fiber is missing. Exception: away from the discriminant locus in $\mathcal{H}$, where the fibers of χ are Abelian varieties, we get the Poincaré bundle for the Fourier–Mukai transform.

Corollary 8.2 *On the complement of the discriminant locus in $\mathcal{H}$, that is, on the part of $\mathcal{H}$ corresponding to smooth spectral curves, $\mathcal{P}$ gives an equivalence of derived categories.*

For details for general G see Donagi–Pantev. For GL, SL, PGL see Hausel–Thaddeus.

Example 8.3 Here are some interesting examples of Fourier transform:

- The structure sheaf of the Hitchin section transforms to $\mathcal{O}$ over $T^*\mathcal{M}$. This follows from the fact that the Poincaré bundle is trivial on the Hitchin section.

- The Fourier transform of $\mathcal{O}$ restricted to $T^*\mathcal{M}^{reg}$ is the structure sheaf of the Hitchin section, shifted in degree $\dim \mathcal{M}$.

Hitchin computed that $R^1\chi_*\mathcal{O}$ is the trivial vector bundle over $\mathcal{H}$ with fibre $\mathcal{H}^\vee$. (So, it works as if χ was a fibre bundle with fibre Abelian varieties. Note that the vertical tangent bundle is isomorphic to $\mathcal{H}^\vee$ via the symplectic structure; and the principal polarisation identifies $\mathcal{H}^\vee$ with $\bar{\mathcal{H}}$ and with H^1 along the fibres.) This implies that $H^1(T^*\mathcal{M}, \mathcal{O}) = \mathbb{C}[\mathcal{H}] \otimes \mathcal{H}^\vee$. This space is also $H^1(\mathcal{M}, Sym\, T)$ and is generated over $H^0(Sym\, T)$ as follows: the polynomial generators of $H^0(Sym\, T)$ are contracted with the first Chern class of the basic line bundle in $H^1(\mathcal{M}, T^*)$ to produce odd generators of $H^1(T^*\mathcal{M}, \mathcal{O})$ over $H^0(Sym\, T)$. Now from the exponential sheaf sequence

$$H^1(T^*\mathcal{M}, \mathbb{Z}) \to H^1(T^*\mathcal{M}, \mathcal{O}) \to H^1(T^*\mathcal{M}, \mathcal{O}^*) \to H^2(T^*\mathcal{M}, \mathbb{Z})$$

we see that H^1 is computing holomorphic structures over the trivial line bundle on $T^*\mathcal{M}$.

Proposition 8.4 *Under Fourier–Mukai transform, the line bundle $\mathcal{O}(\exp H)$ with $H \in \mathcal{H}^\vee$ corresponds to the image of the Hitchin section by* $\exp$ *of the $H-$Hamiltonian flow.*

Note that both objects in this correspondence are analytic and not algebraic. This makes the quantum version more difficult. Insofar as the existing kernel allows us to verify, we have the following equivalence:

Theorem 8.5 *(Frenkel – Teleman)* $H^*(T^*\mathcal{M}, \mathcal{O}) = \mathbb{C}[\mathcal{H}] \otimes \Lambda\mathcal{H}^\vee$.

Here again, equality holds like as if we had a bundle of Abelian varieties. We get also a non-trivial check of *Ext* groups. Denoting by σ the image of the Hitchin section:

Proposition 8.6 $Ext_{T^*\mathcal{M}}(\mathcal{O}_\sigma, \mathcal{O}_\sigma) = \mathbb{C}[\mathcal{H}] \otimes \Lambda\mathcal{H}^\vee$

It seems likely (work in progress) that this implies the following strong result: Fourier–Mukai from from coherent sheaves supported away from $T^*\mathcal{M}^{sing}$ to coherent sheaves on $T^*\mathcal{M}$ is a fully faithful functor.

Proposition 8.7 *There is an equivalence of categories between $Coh(\mathcal{O}_\sigma)$ and the full subcategory of $\mathcal{O}-$modules on $T^*\mathcal{M}$ that are presented by global sections.*

Recall that a sheaf $\mathcal{S}$ is presented by global sections if it is presented as

$$\mathcal{O}^{\oplus p} \to \mathcal{O}^{\oplus q} \to \mathcal{S}.$$

8.2 Deformation of the Hitchin section: Opers

The section $(\oplus K^{-i}, \theta_p)$ with θ_p as in (2) has a deformation corresponding to the Higgs bundle – flat bundle deformation leading to the moduli variety of opers.

Definition 8.8 An *oper* structure on a flat vector bundle E (with group $GL(n)$) is a full flag of subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

satisfying $\nabla(E_i) \subset E_{i+1}$ and moreover such that

$$\nabla \colon E_i/E_{i-1} \tilde{\to} E_{i+1}/E_i \otimes K$$

are isomorphisms.

Remark 8.9 It follows that $E_i/E_{i-1} \simeq E_1/E_0 \otimes K^{\otimes -i+1}$. So, the leading component of ∇ corresponds to the subdiagonal row of 1's in θ_p, see (2).

Proposition 8.10 *(Beilinson–Drinfeld) E carries and oper structure iff it is given by the unique non-trivial successive extension of powers of K. All such bundles carry a holomorphic connection ∇ and the oper structure (the flag) is then unique.*

Note: as a holomorphic bundle E is unstable.

Example 8.11 For $SL(2)$ we must have

$$0 \to K^{1/2} \to E \to K^{-1/2} \to 0$$

and indeed,

$$Ext^1(K^{-1/2}, K^{1/2}) \simeq H^1(K) \simeq \mathbb{C}$$

so the bundle E is unique up to isomorphism. The obstruction to having a holomorphic connection is the Atiyah class in $H^1(\Sigma, K \otimes End(E))$. On the associated graded bundle, $K^{1/2} \oplus K^{-1/2}$ this class is non-trivial. It corresponds to the diagonal class $\begin{bmatrix} g-1 & 0 \\ 0 & 1-g \end{bmatrix}$ in $H^1(\Sigma, \mathrm{diag}(M_2) \otimes K)$. The deformation to non-trivial extensions kills this class and lifts the obstruction to existence of connections.

The space of connections on a fixed bundle E is an affine space over $H^0(End(E) \otimes K)$. If the extension E were split, the latter space would be $H^0(\mathcal{O}) \oplus H^0(K) \oplus H^0(K^{\otimes 2})$; however, in the non-trivial extension, the first summand is disallowed. (It maps isomorphically to $H^1(K)$ in the spectral sequence.) Consequently, the space of oper structures on the fixed bundle E is an affine copy of $H^0(K) \oplus H^0(K^{\otimes 2})$. To get the moduli space of opers, we must divide by the group of automorphisms of E, which is the Abelian group $H^0(K)$ (acting via multiplication by strictly upper-triangular matrices on $Hom(K^{-1/2}, K^{1/2})$. This leads to an affine copy of the Hitchin space.

8.3 Global differential operators on $\mathcal{M}$

This is the non-commutative side of the deformations of $T^*\mathcal{M}$. In the Abelian case, the algebra of differential operators is the free algebra generated by the tangent bundle, because on an Abelian variety global differential operators have constant coefficients. A change in the non-Abelian case is that if G is semi-simple

$$\mathcal{D}(\mathcal{M}) = \Gamma(\mathcal{D}) = \mathbb{C}.$$

In the non-Abelian case, we must consider differential operators on $\sqrt{K}$. Recall that $Pic(\mathcal{M}) = H^2(\mathcal{M}) \simeq \mathbb{Z}$. The class $[\sqrt{K}] = -c \in Pic(\mathbb{Z})$ where c is the dual Coxeter number of G. Denote by $\mathcal{D}_{-c}(\mathcal{M})$ the algebra of differential operators on $\sqrt{K}$.

Theorem 8.12 *(Beilinson–Drinfeld) $\mathcal{D}_{-c}\mathcal{M}$ is the algebra of polynomial functions on opers. The latter is an affine space for $\mathcal{H}$.*

Theorem 8.13 *(Frenkel–Teleman) $H^*(\mathcal{M}, \mathcal{D}_{-c})$ is the polynomial ring of differential forms on opers.*

Beilinson and Drinfeld constructed partial Fourier–Mukai transforms giving

$$\begin{array}{rcl} Coh(Op) & \to & \mathcal{D}_{-c} - modules\ on\ \mathcal{M} \\ point & \mapsto & Hecke\ eigensheaf \end{array}.$$

Their construction is the same one that works on the Abelian case. Recall that $\Gamma(\mathcal{M}, \mathcal{D}_{-c}) = \mathbb{C}[Op]$ and set

$$\widetilde{\mathcal{M}} := \mathcal{D}_{-c} \bigotimes_{\Gamma(\mathcal{M}, \mathcal{D}_{-c})} \Gamma(Op, \mathcal{M}). \tag{3}$$

[BD] proved that $\widetilde{\mathcal{M}}$ is a Hecke eigensheaf.

Example 8.14 In the Abelian case, Op = constant holomorphic 1-forms on A = connection forms on the trivial line bundle. A point in Op is a flat connection = the trivial line bundle, together with the respective flat connection. In 1-dimension, with the connection form α being the constant derivative on $\mathcal{O}$

$$\mathcal{D} \otimes_{\mathbb{C}[\partial]} \mathbb{C}_\alpha = \mathcal{D}/(\partial - \alpha) \simeq \mathcal{O},$$

but with ∂ acting as α.

8.4 Differential operators on $\mathcal{M}$ from the center of $U(L\mathfrak{g})$

If $\mathfrak{g}$ is semi-simple, then $Z(\mathfrak{g}) := Z(U(\mathfrak{g})) = (Sym\,\mathfrak{g})^G$ consists of polynomials in rank $\mathfrak{g}$ generators. The quadratic Casimir is $\Delta_2 = \sum \xi_a \xi_b h^{ab}$ defined using the invariant bilinear form h^{ab}. Moving on to the loop algebra $L\mathfrak{g}$, we note that $U(\mathfrak{g})$ has no center, indeed, the invariant forms would be $\sum_n \xi_a(n)\xi_b(m-n)h^{ab}$ but, at any rate are infinite sum. To keep the algebra structure and allow infinite sums, some restrictions are needed.

We complete $U(L\mathfrak{g})$ by considering its action on highest weight representations. For practical purposes we use the Fourier grading $V = \oplus_{n\leq 0} V(n)$ and $\xi(m)V(n) \subset V(n+m)$. Then sums $\sum_{m>0} \xi_m(m)$ and $\sum_{m>0} \eta_m(k_m)\xi_m(m)$ are allowed. In particular, $L_m = \sum \xi_a(m)\xi_b(m-n)h^{ab}$ is allowed. But it turns out that the infinite sums make the commutation relations nontrivial.

Example 8.15 $[L_m, \xi(n)] = c{\cdot}n\xi(m+n)$ so L_m acts as $z^{m+1}\frac{\partial}{\partial z}$ and generates a copy of the *Virasoro algebra* inside $U(L\mathfrak{g})$.

But, at central extension of $L\mathfrak{g}$ at level c the commutators miraculously vanish and we obtain:

Theorem 8.16 *(Feigen–Frenkel) Elements of $Z(U_{-c}(L\mathfrak{g})^\wedge)$ are functions on the space of opers on the circle.*

From central elements of $U(L\mathfrak{g})$ we can construct differential operators on $\mathcal{M}$ as follows. Let S^1 be a circle bounding a disc Δ on Σ so that $\Sigma = \Sigma^0 \cup_{s_1} \Delta$. Now, use Segal's double coset construction from paragraph 5.3, which gives and isomorphism of analytic stacks from isomorphism classes of G bundles on Σ to double cosets $Hol(\Sigma^0, G)\backslash LG/Hol(\Delta, G)$. This construction has the technical advantage that it can be made algebraic. We can replace LG by the formal loops $G((z))$ in G, also $Hol(\Sigma^0, G)$ by algebraic maps with poles at the centre of Δ, and $Hol(\Delta, G)$ by the formal Taylor loops $G[[z]]$.

Definition 8.17 $\mathbb{X} = G((z))/G[[z]]$ is an algebraic variety with a $G[[z]]-$action, called the *Loop Grassmanian.*

It is an analogue to flag varieties of algebraic groups, the Grassmanians for $GL(n)$. Then $T^*\mathcal{M}$ arises by holomorphic symplectic reduction of $T^*\mathbb{X}$: we have a moment map

$$\mu\colon T^*\mathbb{X} \to (L\mathfrak{g})^* = \mathfrak{g}[[z]]dz$$

for the action of $GL[[z]]$; the image in the coadjoint orbit of $\mathfrak{g}[[z]]dz = T_1^*\mathbb{X}$. Reduction with respect to $G[\Sigma^0]$ intersects the image of μ with $\Gamma(\Sigma^0, \mathfrak{g}\otimes K)$ and divides by $G[\Sigma^0]$. The reduction of $(L\mathfrak{g})^*$ is $\bigoplus_d \Gamma(\Sigma^0, K^{\otimes d})$ and we get a map $T^*\mathcal{M} \to \bigoplus_d \Gamma(\Sigma^0, K^{\otimes d})$, but, in fact, the image lies on the subspace $\bigoplus_d \Gamma(\Sigma, K^{\otimes d})$. The map factors through $\mathcal{H} \to \bigoplus_d \Gamma(\Sigma^0, K^{\otimes d})$ because it was previously in the orbit of $G[[z]]$. The resulting map is the Hitchin morphism. In the quantum analogue, $T^*\mathcal{M} \to \mathcal{D}(\mathcal{M})$ replaces the moment map $T^*\mathbb{X} \to (L\mathfrak{g})^*$ by an algebra morphism

$$U(\mathfrak{g}) \to \Gamma(\mathbb{X}, \mathcal{D})$$

or more generally with a line bundle $\mathcal{O}(h)$ over $\mathbb{X}$, and a map

$$U_h(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_h),$$

where $U_h(L\mathfrak{g})$ is the universal enveloping algebra at level h and $\mathcal{D}_h$ the universal sheaf of differential operators on $\mathcal{O}(h)$. Any central element in $U_h(L\mathfrak{g})$ gives rise to a $G[[z]]-$invariant differential operator on $\mathbb{X}$. Invariance implies that the operators descent to the quotient $\mathcal{M}$. So, we have an algebra homomorphism

$$Z(U_c(L\mathfrak{g})) \to \mathcal{D}_c\mathcal{M},$$

where the subscript $_c$ denotes the central extension at the critical level which corresponds to the dual Coxeter number. But, in fact, just as in the classical case, this factors through restriction from opers in the circle to the finite dimensional subspace of opers on Σ. This is the Beilinson–Drinfeld homomorphism.

9 Geometric Langlands IX

This is the last lecture, which includes a brief review and it is organized as:

1. Motivation for geometric Langlands

2. Translation to characteristic 0 and Kac–Moody algebras. Beilinson-Bernstein correspondence; Lie algebra representations and $\mathcal{D}-$modules. Classical moment map picture and central characters.

3. Loop group analogue of the Beilinson–Bernstein construction. Statement of Feigin–Frenkel theorem comparing the center of $U(L\mathfrak{g})$ to $\mathcal{D}(\mathbb{X})$ in genus 0. Comments on arbitrary genus and relation to representations.

4. Segal double coset construction of $\mathcal{M}$, and $T^*\mathcal{M}$ by symplectic reduction. Moment map and functions on $T^*\mathcal{M}$. Quantization and construction of differential operators from central elements.

0. Two complements on opers:

- The "complete extension" $E^0 \subset E^1 \subset \cdots \subset E^n = E$ with $E^k/E^{k-1} = K^{\otimes -i}$ is the truncated jet bundle of the Riemann surface ($n-$th order jets of functions, dual of differential operators of order $\leq n+1$). The structure group can be reduced to $SL(2)$ (the Riemann surface itself is a quotient of $SL(2,\mathbb{R})$). For general $\mathfrak{g}$ the structure group is defied from the same jet bundle, couple to the principal $SL(2)$ subgroup of G.

- There is a good geometric reason why, on global sections, the only Hecke eigenvalues that are visible correspond to opers (the jet bundle). The global geometry of the Hecke correspondence over the curve ensures that.

  Again, the only Hecke eigen $\mathcal{D}_c-$modules we see through global sections are those corresponding to opers. This is in fact required by the Langlands equivalence and the fact that $\mathcal{D}$ corresponds to the structure sheaf of the subvariety of opers. It $\mathcal{F}$ is a $\mathcal{D}_c-$module, then

  $$\Gamma(\mathcal{M}, \mathcal{F}) = Hom_{\mathcal{D}_c-Mod}(\mathcal{D}_c, \mathcal{F}) = Hom_{M_{flat}}(\mathcal{O}_{op}, \tilde{\mathcal{F}} = 0$$

  if $\tilde{\mathcal{F}}$ does not meet Op.

9.1 Original motivation for Langlands correspondence

The original statement of Langlands, claimed a bijection between certain automorphic representations of $GL(n)$ and certain $n-$dimensional representations of Galois groups. Consider a finite field k and a smooth proper curve Σ over k. For $x \in \Sigma$ we have the local ring $\mathcal{O}_x$ and its completion $\widehat{\mathcal{O}}_x$; e.g. over $\mathbb{C}$, with z a coordinate centered at x we have $\widehat{\mathcal{O}}_x = \mathbb{C}[[z]] \subset \mathbb{C}((z)) = \widehat{k}(\Sigma)$ where $k(\Sigma)$ is the field of rational functions on Σ.

In the automorphic representation side we have the "adèles" $\mathbb{A} = \Pi_{x\in\Sigma}\widehat{K}_x$ where all but finitely many entries are polynomials. We have $\mathbb{O} = \Pi\widehat{\mathcal{O}}_x \subset \mathbb{A}$ the "regular" subrings. We want representations of $GL_n(\mathbb{A}) = \Pi GL_n(\widehat{K})$

which appear in $L^2\left(GL_n(\mathbb{A})/GL_n(k(\Sigma))\right)$. Langlands predicts a bijection between:

$$\begin{array}{c}\{\text{automorphic cuspidal representations of } G(\mathbb{A})\}\\ \Updownarrow \\ \{\text{irreducible representations of } Gal(k(\Sigma)) \text{ into } {}^LG\}.\end{array}$$

Here cuspidal means not induced by a parabolic. Automorphic means occurring in $L^2\left(GL_n(\mathbb{A})/GL_n(k(\Sigma))\right)$. Unramified representations correspond to unramified representations.

Theorem 9.1 *(Weil)* $G(\mathbb{O})\backslash G(\mathbb{A})/G(k(\Sigma)) =$ *set of isomorphism classes of principal G bundles on Σ.*

Geometric Langlands over $\mathbb{C}$ can be motivated in several ways, but one guiding insight is: $G(K_x)$ does not correspond directly to the loop group $G((z))$ of Laurent series in z, but instead to its (Kac–Moody) Lie algebra. Unramified presentations become "vacuum representations" (negative energy) of highest weight $U(\mathfrak{g}) \otimes_{\mathfrak{g}[[z]]} \mathbb{C}$. This insight turns out to be half correct, we do not have the complete formulation rot the moment.

9.2 Bernstein–Beilinson correspondence for reductive group

Let G be a complex reductive group (take $GL(n)$ if you wish), and let $X := G/B$ be the full flag variety (for $GL(n)$ B is the subgroup of upper triangular matrices).

Theorem 9.2 *There is an equivalence between the category of representations of $\mathfrak{g}$ with trivial central character and the category of $\mathcal{D}-$modules on G/B.*

On one side, the map is $M \mapsto \mathcal{D} \otimes_{\mathcal{D}(X)} M$, and the inverse is given by localization. Notes:

1. $\mathfrak{g}$ acts by infinitesimal translations on G/B, giving $\mathfrak{g} \to Vect(X)$. So, it acts on global sections of any $\mathcal{D}-$module.

2. $\mathcal{D}(X) = U(\mathfrak{g}) \otimes_Z \mathbb{C}_0$ where Z is the centre of $U(\mathfrak{g})$, and $\mathbb{C}_0$ is the 1-dimensional module of Z corresponding to the trivial character (the trivial representation of Z).

3. (Harish-Chandra isomorphism) $Z \simeq (Sym\mathfrak{g})^G$ is a polynomial algebra on rank $\mathfrak{g}$ generators, the classical Casimirs. This is the famous Harish-Chandra isomorphism. Note that it is not trivial to produce an isomorphism of algebras, even though it is easy to get a linear map $(Sym\mathfrak{g})^G \to U(\mathfrak{g})^G$ by symmetrization.

4. ($\mathcal{D}-$ affinity of X) If $\mathcal{A}$ is a $\mathcal{D}-$module and $i > 0$ then $R^i\Gamma(\mathcal{A}) = 0$.

5. There is a similar statement as in theorem 9.2 for every central character of $\mathfrak{g}$. For most central characters (dominant ones) it takes the same form, but for certain (regular, but not integrally dominant) characters, one must weaken it to an equivalence of derived categories.

   For singular central characters — the most singular of which corresponds to $\sqrt{K}(-\rho)$ — Γ is an exact functor but not an equivalence: it collapses parts of the category (trivializes a "right action" of the Weyl group on X).

Example 9.3 For $G = SL(2)$ with Casimir operator $\Delta = e_-e_+ + e_+e_- + \frac{1}{2}h$, where

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \; e_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; e_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} ,$$

the center is $Z = \mathbb{C}[\Delta]$.

9.3 The classical picture

$\mathcal{D}$ is the quantization of T^*X. There is a moment map $\mu: T^*X \to \mathfrak{g}^*$ for the translation action of G.

Proposition 9.4 *T^*X maps to the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$.*

Notes:

1. $\mathcal{N}$ is the closure of the $G-$orbit of any regular nilpotent element in $\mathfrak{g}^*$.

2. $\mathcal{N}$ = the zero locus of the positive elements in $(Sym\mathfrak{g})^G \simeq \mathrm{Spec}\mathbb{C}(\mathfrak{g})/I(\Delta_i)$ where Δ_i are the classical Casimirs $\simeq$ the fibre of the quotient map $\mathfrak{g}^* \to \mathfrak{g}^*//G$.

The quantization of this statement is

$$\mathcal{D}(X) = \Gamma(X, \mathcal{D}) = U(\mathfrak{g}/(\Delta_i)$$

and this quantum statement is much better behaved, because there are more vanishing of cohomologies.

Note again that elements of $U(\mathfrak{g})$ give global differential operators on X whereas central elements give left-invariant operators. However since the action of Z factors through the central character, we only get constant (scalar) differential operators from the center.

The classical picture has commutative deformations as well, where we replace $\mathcal{N}$ by a regular semisimple orbit. In the process, T^*X deforms to G/T and μ becomes an isomorphism. Such deformations should be regarded as a variation of the central character in the quantum case, and they morally explain the absence of cohomology: the classical limit is affine.

9.4 Loop groups

The loop group analogue of the Beilinson–Bernstein construction is relevant to the geometric Langlands program. We revisit the Segal double coset construction from 5.3. First some notation:

$$\begin{array}{rcl} LG &=& \text{group of smooth loops in } G \\ G((z)) &=& \text{group of smooth Laurent polynomial loops} \\ G(\Delta) &=& \text{holomorphic maps from the disc } \Delta \text{ to } G \\ G[[z]] &=& \text{group of smooth Taylor polynomial loops} \\ G(\Sigma^*) &=& \text{holomorphic maps from } \Sigma \text{ minus a point to } G \end{array}$$

Theorem 9.5 *(Segal double coset construction) Let $\mathfrak{M}$ be the moduli stack of $G-$bundles on Σ, then*

$$\mathfrak{M} \simeq G[\Sigma^*]\backslash LG/G(\Delta).$$

The objects in $\mathfrak{M}$ are $G[\Sigma^*]-$equivariant objects in $\mathbb{X} = LG/G(\Delta)$ and $G(\Delta)-$equivariant objects in $LG/G[\Sigma^*]$. Classically, we have a moment map for the LG action:

$$\mu\colon T^*\mathbb{X}L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$$

by residue pairing. The image is the $LG-$orbit of $\mathfrak{g}[[z]]dz$ and is a complete intersection defined by the inverse image of the subspace $\bigoplus_d \Gamma(\Delta, K^{\otimes d})$ in

$$L\mathfrak{g}^* \xrightarrow{(\Delta_i)} \bigoplus_d \Gamma(S^1, K^{\otimes d}).$$

However, unlike the finite dimensional case, the map is not a birational equivalence with the image. To see this, consider the deformation of complex structures modulo the LG action. Then

$T^*\mathbb{X} \to \mathfrak{g} -$ connections on the disc

$L\mathfrak{g}^* \to$ connections on the circle with gauge action

The deformed moment map is an isomorphism of LG/G with the orbit of the trivial connection. Note that a connection on the disc is equivalent to an affine line bundle with fibre $\mathfrak{g}[[z]]dz$ over $\mathbb{X}$.

The new image of μ is much bigger, the limiting case has large vertical fibers, in particular imμ has finite codimension. In the limit

$$\mathbb{C}[T^*\mathbb{X}] = \mathbb{C}[L\mathfrak{g}^*] \otimes_{\mathbb{C}[\Gamma_{S^1}]} \mathbb{C}[\Gamma_\Delta] \otimes \mathbb{C}[\Gamma_{S^1}/\Gamma_\Delta].$$