Nekrasov Conjecture for Toric Surfaces

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Instantons on $\mathbb{R}^4$

The statement of the conjecture comes from N. A. Nekrasov, Seiberg-Witten prepotential from instanton counting, and predicts a relation between SUSY $N = 2$ Yang–Mills instanton partition functions and the Seiberg–Witten prepotential.

Field theory description: comes from comparison of the infrared and ultraviolet limits of certain gauge theories. Nekrasov verifies that the vaccum expectation values of their observables is not sensitive to the energy scale.

Nekrasov's Conjecture – informally

1. In the ultraviolet the theory is weakly coupled and dominated by instantons.
2. In the infrared there appears a relation to the prepotential of the effective theory.

Comparing the results of 1 and 2 leads to a conjectural relation between the instanton partition function and the Seiberg–Witten prepotential.

Proofs of the conjecture - $\mathbb{R}^4$ 4d

The conjecture for instantons on $\mathbb{R}^4$ was proven (4d cases)

by Nekrasov and Okounkov in
Seiberg–Witten Theory and Random Partitions – 2003

by Nakajima–Yoshioka in
Instanton Counting on Blowup I. 4-Dimensional Pure Gauge Theory – 2003

by Braverman–Etingof in
Instanton counting via affine Lie algebras II: from Whittaker vectors to the Seiberg–Witten prepotential – 2004

Proofs of the conjecture - $\mathbb{R}^4$ 5d

Note that the papers above address only instantons on $\mathbb{R}^4$. Our result is a proof of a generalised form of Nekrasov's conjecture for instantons on toric surfaces.

Göttsche–Nakajima–Yoshioka (5d theory compactified on a circle)
K-theoretic Donaldson invariants via instanton counting – 2006

Main Theorem - informal version

Theorem (G., Chiu-Chu Melissa Liu)

Nekrasov's conjecture is true for non-compact toric surfaces.

(We prove 8 instances of the conjecture, more later...)

I now explain the formal statement of the theorem for instantons over the open, toric surface

$$\boxed{\Sigma_k^o := \Sigma_k \setminus \ell_\infty = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k)\big)}$$

In this particular example we have the same moduli spaces as the ones considered by Bruzzo – Poghossian – Tanzini in *Instanton counting on Hirzebruch surfaces*– 2008

remark

- Non-compact case: there is a family of theories parametrised by the u-plane.
- Compact case: one integrates over the u-plane, so the partition function depends on one less parameter.

Nekrasov instanton partition function (pure gauge)

$$\boxed{Z_X^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \Lambda) := \sum_{n \geq 0} \Lambda^{2rn} \int_{\mathfrak{M}(X,r,n)} 1}$$

where $\mathfrak{M}(X, r, n)$ is the moduli space of framed $SU(r)$-instantons with charge n on surface X.

Here the ϵ_i are parameters of the small torus action on the surface, and $\vec{a}$ is a vector on the Lie algebra of the big torus that acts on framings.

The integral is taken by formally applying Atiyah–Bott localization and taking the result as the definition.

classical instantons versus SUSY ones

First I intent to compare existence results for classical instantons versus supersymmetric instantons.

Lemma (G., Köppe, Majumdar)

$SU(r)$-instantons on Σ_k^o are in one-to-one correspondence with rank-r holomorphic bundles on Σ_k^o with $c_1 = 0$ together with a framing at infinity.

Gaps on instanton charges

Remark

If $k = 1$, then all holomorphic bundles on Σ_k^o with $c_1 = 0$ correspond to instantons; otherwise there are strong restrictions on the splitting type.

In particular, this implies gaps on the values of the topological charge.

Theorem (G., Köppe, Majumdar)

If E is a nontrivial $SU(2)$-instanton on Σ_k^o, then its charge satisfies

$$\chi(E) \geq k - 1.$$

Local instanton charge

Definition

Let $\pi\colon \tilde{X} \to X$ be a resolution of a singularity $x \in X$ and E sheaf on $\tilde{X}$ then the local charge of E is

$$\chi^{loc}(E) := l(R^1\pi_*E) + l\left(\frac{(\pi_*E)^{\vee\vee}}{\pi_*E}\right) = \mathbf{h}_k(E) + \mathbf{w}_k(E)$$

Gaps on instanton charges

Theorem (Ballico, Köppe, G.)

Let E be an algebraic rank-2 vector bundle over Σ_k^0 with $c_1 = 0$ and splitting type $j > 0$. Let $n_1 = \left\lfloor \frac{j-2}{k} \right\rfloor$ and $n_2 = \left\lfloor \frac{j}{k} \right\rfloor$. Then the following bounds are sharp:

$$j - 1 \leq \mathbf{h}_k(E) \leq (j-1)(n_1+1) - k\binom{n_1}{2}$$

$$0 \leq \mathbf{w}_k(E) \leq (j+1)n_2 - k\binom{n_2}{2}$$

and $\mathbf{w}_1(E) \geq 1$.

(????)

Back to SUSY instantons

In greater generality, for framed, torsion-free sheaves of degree d and $c_2 = n$ we have:

Lemma

$\mathfrak{M}(\Sigma_k, r, d, n)$ is smooth of dimension $2nr + k(r-1)d^2$.

Computation technique: We use the Atiyah–Bott Localisation Theorem for the toric action on the moduli spaces to compute Nekrasov's partition function.

Torus action on the moduli spaces

We have a torus $\widetilde{T} = T_t \times T_e$ where:
the small torus $T_t \simeq \mathbb{C}^* \times \mathbb{C}^*$ acts on the surface X,
the big torus T_e is the maximal torus of $GL(r)$ acting on frames.

- For $(t_1, t_2) \in T_t$ we denote by $F_{(t_1,t_2)}$ the automorphism of X given by $F_{(t_1,t_2)}(x) = (t_1, t_2) \cdot x$.
- For $\vec{e} = \mathrm{diag}(e_1, \ldots, e_r) \in T_e$ we denote by $G_{\vec{e}}$ the isomorphism of $\mathcal{O}_{\ell_\infty}^{\oplus r}$ given by $(s_1, \ldots, s_r) \mapsto (e_1 s_1, \ldots, e_r s_r)$.

Torus action on the moduli spaces - cont.

- The above actions induce an action of $\widetilde{T}$ on the moduli space: given $(E,\Phi) \in \mathfrak{M}_{r,d,n}(X,\ell_\infty)$ set

$$(t_1,t_2,\vec{e}) \cdot (E,\Phi) = \left((F_{t_1,t_2}^{-1})^* E, \Phi'\right) ,$$

where Φ' is the composition of homomorphisms

$$(F_{t_1,t_2}^{-1})^* E|_{\ell_\infty} \xrightarrow{(F_{t_1,t_2}^{-1})^*\Phi} (F_{t_1,t_2}^{-1})^* \mathcal{O}_{\ell_\infty}^{\oplus r} \longrightarrow \mathcal{O}_{\ell_\infty}^{\oplus r} \xrightarrow{G_e} \mathcal{O}_{\ell_\infty}^{\oplus r} .$$

Fixed point set I

The set of fixed points $\mathfrak{M}_{r,d,n}(X,\ell_\infty)^{\widetilde{T}}$ consist of

$$(E,\Phi) = \big(I_1(D_1),\Phi_1\big) \oplus \cdots \oplus \big(I_2(D_r),\Phi_r\big)$$

such that

- $I_\alpha(D_\alpha) = I_\alpha \otimes \mathcal{O}_X(D_\alpha)$
  - D_α is a T_t-invariant divisor in $X_0 = X \setminus \ell_\infty$
  - I_α are ideal sheaves of 0-dimensional subschemes Q_α in X_0.
- I_α is fixed by the action of T_t.
- Φ_α is an isomorphism from $(I_\alpha)_{\ell_\infty}$ to the αth factor of $\mathcal{O}_{\ell_\infty}^{\oplus r}$.

The support of Q_α must be contained in the fixed point set in X.

Fixed point set II

Each Q_α is a union of subschemes Q_α^v supported at a fixed point $p_v \in X_0$. If we take a coordinate system (x, y) around p_v then the ideal of Q_α^v is generated by monomials $x^i y^j$, so Q_α^v corresponds to a Young diagram Y_α^v.

Relation to the surface X^0: we write a graph Γ so that the fixed point set gets described in terms of combinatorial data:

$$\begin{aligned} \{\text{vertices of } \Gamma\} \quad &\Longleftrightarrow \quad \left\{ T_t\text{-fixed points in } X^0 \right\} \\ \{\text{edge of } \Gamma\} \quad &\Longleftrightarrow \quad \left\{ T_t\text{-invariant } \mathbb{P}^1 \text{ in } X^0 \right\} \end{aligned}$$

Equivariant cohomology

Let $ET \to BT$ be the universal T-bundle. Then

$$BT = B(\mathbb{C}^*)^{2+r} \cong (B\mathbb{C}^*)^{2+r}$$

where $B\mathbb{C}^* = \mathbb{P}^\infty$. The T-equivariant cohomology of X is

$$H_T^*(X) = H^*(ET \times_T X)$$

is the homotopy orbit space. In particular,

$$H_T^*(\mathrm{pt}) = H_T^*(BT) \cong \mathbb{Q}[\epsilon_1, \epsilon_2, a_1, \ldots, a_r].$$

Atiyah–Bott Localization

Notation:

- M^T = T-fixed points of M.
- N = normal bundle of M^T.

$$\int_M \alpha = \int_{M^T} \frac{i^*\alpha}{e_T(N)} ,$$

where $i\colon M^T \hookrightarrow M$ is the inclusion.

In particular, when M^T consists of isolated points, then

$$\int_M \alpha = \sum_{j=1}^{N} \frac{i^*\alpha}{e_T(T_{p_j}M)} .$$

Equivariant parameters

For $i=1,2$, let $p_i\colon BT_t \cong \mathbb{P}^\infty \times \mathbb{P}^\infty$ be the projection to the i-th factor, and let $\epsilon_i = (c_1)_{T_t}(p_i^*\mathcal{O}(1))$. Then

$$H^*_{T_t}(pt;\mathbb{Q}) = H^*(BT_t;\mathbb{Q}) = \mathbb{Q}[\epsilon_1,\epsilon_2].$$

Let $t_i = e^{\epsilon_i} = \mathrm{ch}_1(p_i^*\mathcal{O}(1))$.
Similarly, for $j=1,\ldots,r$, let $q_j : BT_e \cong (\mathbb{P}^\infty)^r \to \mathbb{P}^\infty$ be the projection to the j-th factor, and let $a_j = (c_1)_{T_t}(q_j^*\mathcal{O}(1))$. Then

$$H^*_{T_e}(pt;\mathbb{Q}) = H^*(BT_e;\mathbb{Q}) = \mathbb{Q}[a_1,\ldots,a_r].$$

Let $e_j = e^{a_j} = \mathrm{ch}_1(q_j^*\mathcal{O}(1))$. We write $\vec{a} = (a_1,\ldots,a_r)$ and $\vec{e} = (e_1,\ldots,e_r) = (e^{a_1},\ldots,e^{a_r})$.

Example

$T_t = C^* \times C^*$ acting on $\mathbb{P}^2$ by

$$(t_1, t_2)[z_0, z_1, z_2] = [z_0, t_1 z_1, t_2 z_2]$$

here we have $H^*_{T_t}(pt, \mathbb{Q}) = \mathbb{Q}[\epsilon_1, \epsilon_2]$ and

$$\int_{\mathbb{P}^2} 1 = \frac{1}{\epsilon_1\epsilon_2} + \frac{1}{(-\epsilon_1)(-\epsilon_1+\epsilon_2)} + \frac{1}{(-\epsilon_2)(\epsilon_1-\epsilon_2)} = 0$$

$$\int_{\mathbb{C}^2} 1 = \frac{1}{\epsilon_1\epsilon_2}$$

Main Theorem - Pure Gauge Theory case (Definitions)

ϵ_1, ϵ_2 = weights of the small torus action

$\vec{a}$ = vector in the Lie algebra of the big torus

Λ = formal variable

Definition

$$F^{\text{inst}}_{\Sigma_k,d}(\epsilon_1, \epsilon_2, \vec{a}; \Lambda) := \log Z^{\text{inst}}_{\Sigma_k,d}(\epsilon_1, \epsilon_2, \vec{a}; \Lambda) \ .$$

Theorem (G., Chiu-Chu Melissa Liu)

Statement for Σ_k^0

- *The function*

$$\epsilon_2(k\epsilon_1+\epsilon_2)\ F^{inst}_{\Sigma_k^0,d}(\epsilon_1,\epsilon_2,\vec{a};\Lambda)$$

is analytic in ϵ_1, ϵ_2 near $\epsilon_1 = \epsilon_2 = 0$.

- *The limit at zero is*

$$\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_2(k\epsilon_1+\epsilon_2)\ F^{inst}_{\Sigma_k^0,d}(\epsilon_1,\epsilon_2,\vec{a};\Lambda)=k\mathcal{F}_0^{inst}(\vec{a};\Lambda)\ ,$$

where $\mathcal{F}_0^{inst}(\vec{a};\Lambda)$ is the instanton part of the Seiberg–Witten prepotential.

Expression for Nekrasov's partition function

$$Z^{\text{inst}}_{\Sigma^0_k,d}(\epsilon_1,\epsilon_2,\vec{a};\Lambda) = \sum_{\{\vec{d}\}=-\frac{d}{r}} \frac{\Lambda^{kr(\vec{d},\vec{d})}}{\prod_{\alpha,\beta} l^{k,\vec{d}}_{\alpha,\beta}(\epsilon_1,\epsilon_2,\vec{a})} \cdot$$

$$Z^{\text{inst}}_{\mathbb{C}^2}(\epsilon_1,\epsilon_2,\vec{a}+\epsilon_2\vec{d};\Lambda)\cdot Z^{\text{inst}}_{\mathbb{C}^2}(-\epsilon_1,k\epsilon_1+\epsilon_2,\vec{a}+(k\epsilon_1+\epsilon_2)\vec{d};\Lambda)\ ,$$

where we used expression of the partition function for $\mathbb{C}^2$

$$Z^{\text{inst}}_{\mathbb{C}^2}(\epsilon_1,\epsilon_2,\vec{a};\Lambda) = \sum_{\vec{Y}} \frac{\Lambda^{2r|\vec{Y}|}}{\prod_{\alpha,\beta=1}^{r} n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1,\epsilon_2,\vec{a})}\ , \qquad \text{and...}$$

. . . and

$$l^{k,\vec{d}}_{\alpha,\beta}(\epsilon_1,\epsilon_2,\vec{a}) = \begin{cases} \prod_{j=0}^{d_\alpha-d_\beta-1}\prod_{i=0}^{kj}(-i\epsilon_1 - j\epsilon_2 + a_\beta - a_\alpha) & \text{if } d_\alpha > d_\beta, \\ \prod_{j=1}^{d_\beta-d_\alpha}\prod_{i=1}^{kj-1}(i\epsilon_1 + j\epsilon_2 + a_\beta - a_\alpha) & \text{if } d_\alpha < d_\beta, \\ 1 & \text{if } d_\alpha = d_\beta, \end{cases}$$

$$n^{\vec{Y}}_{\alpha,\beta}(t_1,t_2) = \prod_{s\in Y_\alpha}\Big(-l_{Y_\beta(s)}\epsilon_1 + (a_{Y_\alpha}(s)+1)\epsilon_2 + a_\beta - a_\alpha\Big) \cdot \prod_{t\in Y_\beta}\Big((l_{Y_\alpha(t)}+1)\epsilon_1 - a_{Y_\beta(t)}\epsilon_2 + a_\beta - a_\alpha\Big) .$$

Susy

The Seiberg–Witten Prepotential (Definition Part 1)

Seiberg–Witten curves are a family of hyperelliptic curves parametrised by $\vec{u} = (u_2, \cdots, u_r)$:

$$C_{\vec{u}} : \Lambda^r \left(w + \frac{1}{w} \right) = P(z) = z^r + u_2 z^{r-2} + u_3 z^{r-3} + \cdots + u_r \ ,$$

coming together with the double cover $C_{\vec{u}} \to \mathbb{P}^1$ given by the projection $(w, z) \mapsto z$.

The parameter space $\vec{u} \in \mathbb{C}^{r-1}$ is the so-called u-plane.

The Seiberg–Witten Prepotential (Definition Part 2)

The hyperelliptic involution is given by $\iota(w) = 1/w$.

Introducing $y = \Lambda^r \left(w - \frac{1}{w}\right)$ we have

$$y^2 = P(z)^2 - 4\Lambda^{2r}.$$

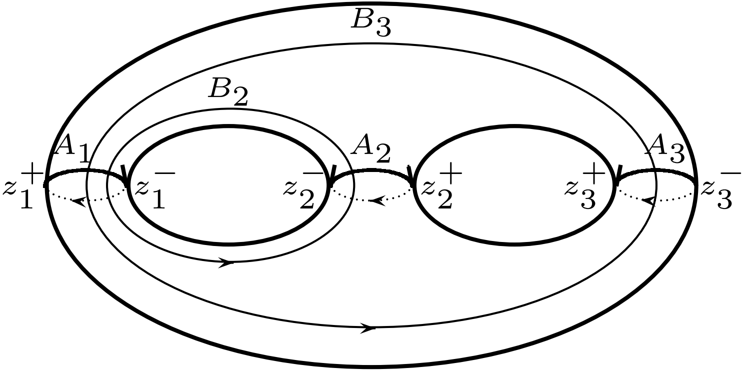
We choose a symplectic basis $\{A_\alpha, B_\alpha, \alpha = 2, \ldots, r\}$ of $H_1(C_{\vec{u}}; \mathbb{Z})$; consequently

$$A_\alpha \cdot A_\beta = 0 = B_\alpha \cdot B_\beta$$

and

$$A_\alpha \cdot B_\beta = \delta_{\alpha\beta} \ .$$

Symplectic basis on $C_{\vec{u}}$



The Seiberg–Witten Prepotential (Definition Part 3)

The (multivalued meromorphic) Seiberg–Witten differential is defined by

$$dS := -\frac{1}{2pi}\frac{dw}{w} = -\frac{1}{2\pi}\frac{zP'(z)dz}{\sqrt{P(z)^2 - 4\Lambda^{2r}}} \ .$$

The Seiberg–Witten Prepotential (Definition Part 4)

Functions a_α, a_β^D on the u-plane are defined by

$$a_\alpha := \int_{A_\alpha} dS \ , \quad a_\beta^D := 2\pi\sqrt{-1}\int_{B_\beta} dS$$

for $\alpha = 1, \ldots, r$ and $\beta = 2, \ldots, r$.

The Seiberg–Witten Prepotential (Finally!)

The Seiberg–Witten prepotential is a locally defined function $\mathcal{F}_0(\vec{a};\Lambda)$ on the $\vec{u}$-plane satisfying:

$$a_\alpha^D = -\frac{\partial \mathcal{F}_0}{\partial a_\alpha} \ .$$

Periods

It then follows that

$$\tau_{\alpha\beta} = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 \mathcal{F}_0}{\partial a_\alpha \partial a_\beta}$$

is the period matrix of $C_{\vec{u}}$.

(Defined in *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ Supersymetric Yang Mills theory*.)

The instanton partition functions in $4D$, $5D$ and $6D$ correspond to the generating series of the holomorphic Euler characteristic χ_0, the Hirzebruch genus χ_y, and elliptic genus χ of $\mathfrak{M}(\Sigma_k, 1, d, n)$, respectively.

The 4D case: Holomorphic Euler Characteristic: $\chi(\mathcal{O}_{\mathfrak{M}})$

$$Z_{\mathbb{C}^2}(t_1, t_2; Q) = \sum_Y \frac{Q^{|Y|}}{\prod_{s\in Y}\left(1 - t_1^{l(s)} t_2^{-1-a(s)}\right)\left(1 - t_1^{-1-l(s)} t_2^{a(s)}\right)}$$

$$Z_{\Sigma_k,d}(t_1, t_2; Q) = Z_{\mathbb{C}^2}(t_1, t_2; Q)\ Z_{\mathbb{C}^2}(t_1^{-1}, t_1^k t_2; Q)\ .$$

The 5D case: Hirzebruch $\chi_{\mathbf{y}}$-genus

$$Z_{\mathbb{C}^2}(t_1, t_2; Q, y) = \sum_{Y} Q^{|Y|} \prod_{s \in Y^1} \frac{\left(1 - y t_1^{l(s)} t_2^{-1-a(s)}\right)\left(1 - y t_1^{-1-l(s)} t_2^{a(s)}\right)}{\left(1 - t_1^{l(s)} t_2^{-1-a(s)}\right)\left(1 - t_1^{-1-l(s)} t_2^{a(s)}\right)}$$

$$Z_{\Sigma_k, d}(t_1, t_2; Q, y) = Z_{\mathbb{C}^2}(t_1, t_2; Q, y)\ Z_{\mathbb{C}^2}(t_1^{-1}, t_1^k t_2; Q, y)\ .$$

The 6D case: Elliptic genus

$$Z_{\mathbb{C}^2}(t_1, t_2; Q, y, p) = \sum_Y (y^{-1}Q)^{|Y|} \prod_{n\geq 1}$$

$$\prod_{s\in Y} \frac{\left(1 - yp^{n-1} t_1^{l(s)} t_2^{-1-a(s)}\right)\left(1 - y^{-1} p^n t_1^{-l(s)} t_2^{1+a(s)}\right)}{\left(1 - p^{n-1} t_1^{l(s)} t_2^{-1-a(s)}\right)\left(1 - p^n t_1^{-l(s)} t_2^{1+a(s)}\right)} \cdot$$

$$\prod_{s\in Y} \frac{\left(1 - yp^{n-1} t_1^{-1-l(s)} t_2^{a(s)}\right)\left(1 - y^{-1} p^n t_1^{1+l(s)} t_2^{-a(s)}\right)}{\left(1 - p^{n-1} t_1^{-1-l(s)} t_2^{a(s)}\right)\left(1 - p^n t_1^{1+l(s)} t_2^{-a(s)}\right)}$$

and

$$Z_{\Sigma_k,d}(t_1, t_2; Q, y, p) = Z_{\mathbb{C}^2}(t_1, t_2; Q, y, p)\ Z_{\mathbb{C}^2}(t_1^{-1}, t_1^k t_2; Q, y, p)\ .$$

General form of the partition function

Given two multiplicative classes A, B we define

$$\boxed{Z^{\text{inst}}_{X_0,A,B,d}(\epsilon_1,\epsilon_2,\vec{a};\Lambda) := \Lambda^{(1-r)d\cdot d}\sum_{n\geq 0}\Lambda^{2rn}\int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} A_{\tilde{t}}(T_{\mathfrak{M}})B_{\tilde{t}}(V)}$$

where $T_{\mathfrak{M}} =$ tangent bundle and $V =$ natural bundle on $\mathfrak{M}$ obtained from the universal sheaf $\mathcal{E} \to X \times \mathfrak{M}$. Let p_i be the projections to the two factors.

The natural bundle over $\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ is

$$V := (R^1 p_2)_*(\mathcal{E} \otimes p_1^*(\mathcal{O}_X(-\ell_\infty))).$$

Main Theorem - Prototype statement

Theorem

Nekrasov conjecture for toric surfaces

(a) $\mathcal{F}^{\cdots}_{X_0,A,B,d}(\epsilon_1,\epsilon_2,\vec{a},\mathbf{m};\Lambda)$ *is analytic in* ϵ_1,ϵ_2 *near* $\epsilon_1=\epsilon_2=0$.

(b) $\lim\limits_{\epsilon_1,\epsilon_2\to 0}\mathcal{F}^{\cdots}_{X_0,A,B,d}(\epsilon_1,\epsilon_2,\vec{a};\Lambda)=k\mathcal{F}^{\cdots}_0(\vec{a},\Lambda)$, *where* $\mathcal{F}^{\cdots}_0(\vec{a},\Lambda)$ *is the* $\cdots$*part of the Seiberg-Witten prepotential of matter case* $A,B,\mathbf{m}$, *and* $k=\ell_\infty\cdot\ell_\infty>0$ *is the self intersection number of* ℓ_∞.

The 8 cases we prove are:

Instanton part

With the $\cdots$ replaced by $^{\text{inst}}$

1. 4d pure gauge theory: $A = B = 1$, $\mathbf{m} = \emptyset$.
2. 4d gauge theory with N_f fundamental matter hypermultiplets: $A = 1$, $B = e_{\mathbf{T_m}}(V \otimes M)$, $\mathbf{m} = (m_1, \ldots, m_{N_f})$, where
   M is the fundamental representation of $U(N_f)$
   T_m is the maximal torus of $U(N_f)$
3. 4d gauge theory with one adjoint matter hypermultiplet: $A = e_m T_{\mathfrak{M}}$, $B = 1$, $\mathbf{m} = m$.
4. 5d gauge theory compactified on a circle: $A = \hat{A}_\beta(T_{\mathfrak{M}})$ is the $\hat{A}_\beta$ genus of the tangent bundle (the usual $\hat{A}$ genus being the case $\beta = 1$), $B = 1$, $m = \emptyset$ but F depends on the additional parameter β.

Perturbative part

With the $^{\cdots}$ replaced by $^{\text{pert}}$, we derive 4 more cases of the conjecture, with same restrictions as in the first part:

1. 4d pure gauge theory.
2. 4d gauge theory with N_f fundamental matter hypermultiplets.
3. 4d gauge theory with one adjoint matter hypermultiplet.
4. 5d gauge theory compactified on a circle or circumference β.

Obrigada pela atenção!