A First Lecture on Sheaf Cohomology

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I. THE DEFINITION OF SHEAF.

The easiest way to understand the definition of sheaf is to first recall the properties of sections of vector bundles. A sheaf is made up of elements which behave like the sections of a vector bundle. However, even before doing this one can pose the question of why would it be interesting to look at sections of vector bundles. There are many reasons for that and maybe the most important of them is the fact that solutions of differential equations over a manifold are sections of vector bundles over this manifold.

It is important to note also that a vector bundle is completely determined by its set of local sections. The concept of sheaf is more general than the concept of bundle, in the sense that the set of local sections of a vector bundle forms a sheaf, but there are sheaves which do not correspond to sections of a bundle. One big advantage of working with sheaves is the fact that they can be used as coefficients for cohomologies and in fact our aim here is to define sheaf cohomology and give examples.

Let us now look closely at sections of a vector bundle. Let E be a vector bundle over a manifold M with projection map $\pi : E \to M$.

Definition: A *section* of the vector bundle E is a function $\sigma : M \to E$ satisfying $\sigma(x) \in \pi^{-1}(x)$ for all $x \in M$. That is, for every point x in M, $\sigma(x)$ belongs to the fiber over x; or in other words, a section picks out one point on every fiber of the bundle. If $U \subset M$ is an open subset of M, then a section of $E|_U$ (= the bundle E restricted to U) is called a *local section* of E.

If we now look at the behavior of local sections of E, the following 3 properties are evident:
1. Let $W \subset V \subset U$ be open subsets of M. If σ_U is a local section of E defined over U, then restricting σ_U to V and then restricting the result again to W gives us the same as restricting σ_U to W at once.

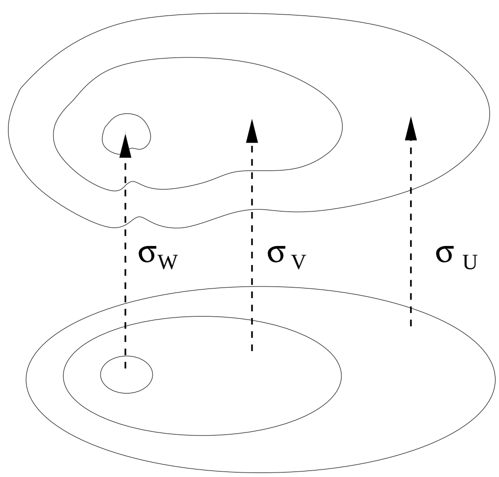

fig. 1.

2. Let U and V be two open subsets of M and let σ_U be a local section defined over U and σ_V a local section defined over V which coincide in the overlaping $U \cap V$. Then there exists a section ρ defined over the union $U \cup V$ which coincides with σ_U over U and coincides with σ_V over V.

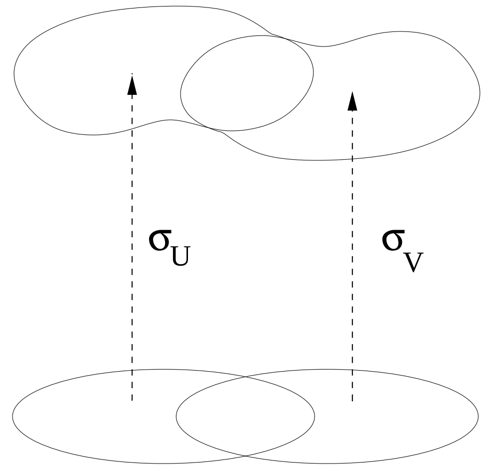

fig. 2.

3. Let U and V be two open subsets of M and let σ be a local section defined over the union $U \cup V$ such that σ vanishes over U and σ vanishes over V, then σ vanishes over the union $U \cup V$.

We are now ready for the definition of sheaf, which follows copied from Griffths and Harris [1] pg. 35.

Definition: Given M a topological space, a *sheaf* $\mathcal{F}$ on M associates to each open set $U \subset M$ a group $\mathcal{F}(U)$, called the *sections* of $\mathcal{F}$ over U, and to each pair $U \subset V$ of open sets a map $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, called the *restriction* map, satisfying

1. For any triple $W \subset V \subset U$ of open sets, $r_{U,W} = r_{V,W} \cdot r_{U,V}$, where $\cdot$ means composition.
2. For any pair of open sets $U, V \subset M$ and sections $\sigma \in \mathcal{F}(U), \tau \in \mathcal{F}(V)$ such that $\sigma|_{U\cap V} = \tau|_{U\cap V}$ there exists a section $\rho \in \mathcal{F}(U \cup V)$ with $\rho|_U = \sigma$ and $\rho|_V = \tau$.
3. If $\sigma \in \mathcal{F}(U \cup V)$ and $\sigma|_U = \sigma|_V = 0$ then $\sigma = 0$.

Examples:

1. The sheaf $\mathcal{C}^\infty$ of local C^∞ functions on M, that to an open set $U \subset M$ the the group $\mathcal{C}^\infty(U)$ of C^∞ functions on U. Here $\mathcal{C}^\infty(U)$ is a group with pointwise addition of functions.

2. The sheaf $\mathcal{C}^*$ of nonzero C^∞ functions on M, that to an open set $U \subset M$ associates the group $\mathcal{C}^*(U)$ of nonzero (meaning not zero at any point) C^∞ functions on U. Here $\mathcal{C}^*(U)$ is a group with pointwise multiplication of functions.

3. The sheaf $\mathcal{O}$ of local holomorphic functions on M, that to an open set $U \subset M$ associates the group $\mathcal{O}(U)$ of holomorphic functions on U. Here $\mathcal{O}(U)$ is a group with addition of functions.

4. The sheaf $\mathcal{O}^*$ of nonzero local holomorphic functions on M, that to an open set $U \subset M$ the group $\mathcal{O}^*(U)$ associates the nonzero holomorphic functions on U. Here $\mathcal{O}^*(U)$ is a group with multiplication of functions.

5. Let E be holomorphic vector bundle over a complex manifold M. The sheaf $\mathcal{O}(E)$ of local holomorphic sections of E assigns to an open set $U \subset M$ associates the group $\mathcal{O}(E)(U)$ of local holomorphic sections of E defined over U. Here the group operation for $\mathcal{O}(E)(U)$ is pointwise addition of sections.

6. Let V be a subvariety of a complex manifold M. The sheaf $\mathcal{I}_V$ (called the *ideal sheaf of* V) formed by the local holomorphic functions vanishing on V, assigns to an open set $U \subset M$ the group $\mathcal{I}_V(U)$ of holomorphic functions on U vanishing on $U \cap V$. We see $\mathcal{I}_V(U)$ as a group with the operation of sum of functions, and moreover it forms an ideal over $\mathcal{O}(U)$ with pointwise multiplication, hence the name ideal sheaf.

7. The sheaf $\mathcal{A}^p$ of C^∞ p-forms on M, that to an open set $U \subset M$ associates the group $\mathcal{A}^p(U)$ of smooth p-forms over U. Here the group operation for $\mathcal{A}^p(U)$ is pointwise addition.

8. The sheaf $\mathbf{Z}$ (or $\mathbf{Q}$ or $\mathbf{R}$ or $\mathbf{C}$) of locally constant $\mathbf{Z}$-valued (or $\mathbf{Q}$ or $\mathbf{R}$ or $\mathbf{C}$-valued) functions on M that to an open set $U \subset M$ associates the group $\mathbf{Z}(U)$ (or $\mathbf{Q}(U)$ or $\mathbf{R}(U)$ or $\mathbf{C}(U)$) of constant $\mathbf{Z}$-valued (or $\mathbf{Q}$ or $\mathbf{R}$ or $\mathbf{C}$-valued) functions over U. Here the group operation is addition.

Exercise 1. Let $\mathcal{A}^{p,q}$ denote the local C^∞ forms of type (p,q) over the complex manifold M, that is the forms that are represented locally in the form

$$\sum \phi_{i_1,\cdots,i_p,j_1,\cdots,j_p} z^{i_1}\cdots z^{i_p}\bar{z}^{j_1}\cdots \bar{z}^{j_q}.$$

Show that $\mathcal{A}^{p,q}$ is a sheaf over M.

Exercise 2. Show that $\mathcal{O}(\mathbf{C}^2-\{0\})=\mathcal{O}(\mathbf{C}^2)$. Hint: Use Hartog's theorem.

Definition: A *map of sheaves* $\alpha:\mathcal{F}\to\mathcal{G}$ on M is given by a collection of local maps $\alpha_U:\mathcal{F}(U)\to\mathcal{G}(U)$ which are group homomorphisms and such that for open subsets $U\subset V\subset M$, α_U and α_V commute with the restriction maps.

Exercise 3. Given a map of sheaves $\alpha:\mathcal{F}\to\mathcal{G}$ on M define $\mathbf{K}er(\alpha)$ by taking the kernel of α on the open subsets of M, that is, $\mathbf{K}er(\alpha)(U)=Ker(\alpha_U:\mathcal{F}(U)\to\mathcal{G}(U))$. Show that $\mathbf{K}er\,(\alpha)$ is a sheaf over M.

Exercise 4. Consider the sheaf map $exp:\mathcal{O}\to\mathcal{O}^*$ over $\mathbf{C}-\{0\}$ given by sending $f\in\mathcal{O}(U)$ to $e^{2\pi i f}\in\mathcal{O}^*(U)$. Show that the section $z\in\mathcal{O}^*(\mathbf{C}-\{0\})$ is not in the image of exp, but its restriction to any contractible open set $U\subset\mathbf{C}-\{0\}$ is in the image of $\mathcal{O}(U)$.

Exercise 5. Given a map of sheaves $\alpha:\mathcal{F}\to\mathcal{G}$ on M define $\mathbf{I}m(\alpha)$ by taking the image of α on the open subsets of M, that is, by taking $U\mapsto im\,\alpha(U)$. Show that $\mathbf{I}m\,(\alpha)$ is *not* a sheaf over M. (In fact this is an example of a pre-sheaf (see [2]) that is not a sheaf.)

Definition: Let $\mathcal{F}$ be a sheaf on M and let $x\in U\cup V\subset M$. We say that two sections α_U and α_V belong to the same *germ* of $\mathcal{F}$ at x if they agree on a neighborhood of x, that is, if there exists an open set W with $x\in W\subset U\cap V$ such that $\alpha_U|_W=\alpha_V|_W$. The collection of all germs of $\mathcal{F}$ at x is called the *stalk* of $\mathcal{F}$ at x and is denoted by $\mathcal{F}_x$.

Remark: We remark that germs and stalks can be defined in more generality, even without requiring that $\mathcal{F}$ be a sheaf, since the properties defining a sheaf are not used to define the germs. In particular we may talk of the stalks of $\mathbf{I}m(\alpha)$ from exercise 5. Now, given a sheaf map, we certainly want to have the property that the image of a sheaf by a sheaf map be again a sheaf. In order to have this some extra work is needed on what we had in exercise 4, this construction is what is known as sheafification and is explained in exercise 6.

Exercise 6. Let $\mathbf{I}m(\alpha)$ be defined as in exercise 4 and construct $\mathbf{I}m^+(\alpha)$ as follows. For any open set $U\subset M$, let $\mathbf{I}m^+(\alpha)(U)$ be the set of functions $s:U\to\bigcup_{x\in U}\mathbf{I}m(\alpha)_x$ which take values on the stalks of $\mathbf{I}m(\alpha)$ and agree with the sections of $\mathbf{I}m(\alpha)$ on small neighborhoods, namely:

$i)s(x)\in\mathbf{I}m(\alpha)_x$ for each $x\in U$

$ii)$ for each $x\in U$ there is an open neighborhood $x\in V\subset U$ and an element $t\in\mathbf{I}m(\alpha)(V)$ such that for all $y\in V$ theX germ $t(y)$ equals $s(y)$.
Show that $\mathbf{I}m^+(\alpha)(U)$ is a sheaf over M. It is called the *image* of α.

II. SHEAF COHOMOLOGY

To define sheaf cohomology we go through the same steps as for defining the usual cohomologies (eg. singular cohomology, de Rham cohomology) namely, we define cochains and a coboundary map δ satisfying $\delta^2=0$. Once this is done, cohomology groups are defined by quotients $\{cocycles\}/\{coboundaries\}$ as usual.

First of all let us fix a locally finite open cover $\mathcal{U}=\{U_\alpha\}$ of M. In order to define the cohomology groups $\check{H}^*(\mathcal{U},\mathcal{F})$ we proceed as follows. The 0-cochains are by definition sections of $\mathcal{F}$ defined over the open sets U_α (see fig. 3.). A 1-cochain is a section of $\mathcal{F}$ defined over double intersections $U_\alpha\cap U_\beta$, (see fig. 4.) and in general an n-cochain is a section of $\mathcal{F}$ defined over the intersections of $n+1$ open sets $U_{\alpha_0}\cap\cdots\cap U_{\alpha_n}$. The set of all n-cochains is denoted by $C^n(\mathcal{U},\mathcal{F})$.

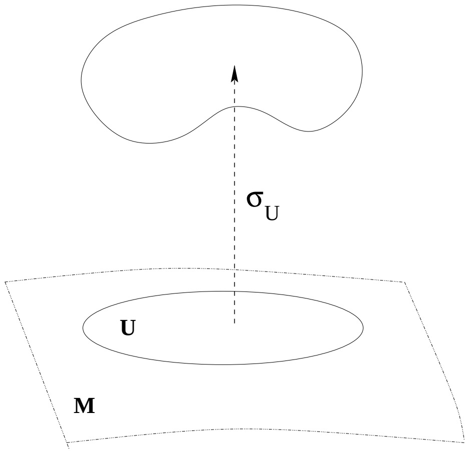

fig. 3.

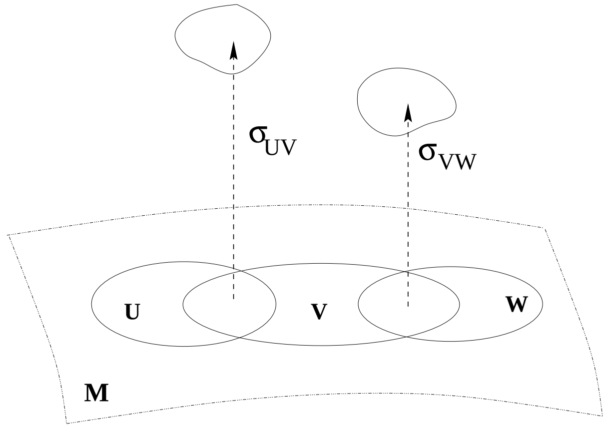

fig. 4.

The formal definition, as stated in Griffths and Harris [1] pg. 38 goes as:

Definition: Let $\mathcal{F}$ be a sheaf on M and $\mathcal{U} = \{U_\alpha\}$ a locally finite open cover. We define

$$C^n(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0 \neq \alpha_1 \neq \cdots \neq \alpha_n} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}).$$

An element $\sigma = \{\sigma_I \in \mathcal{F}(\cap U_{i_k})_{\#I=n+1}\}$ of $C^n(\mathcal{U}, \mathcal{F})$ is called an *n-cochain* of $\mathcal{F}$.

As usual for cohomology theories, the next step is to define the coboundary operator. Such operator δ must raise by one the order of the cochains, taking an n-cochain to an $n+1$-cochain. In our case we must have $\delta : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F})$.

Given a 0-cochain $\sigma = \{\sigma_U\}$ its coboundary $\delta\sigma$ will be a 1-cochain and hence will be defined over the double intersections $U \cap V$. The intuitive meaning of $\delta\sigma_{U,V}$ is that it measures "how far" σ_U and σ_V are from defining one single section over the union $U \cup V$. In case $\sigma_U|_{U\cap V} = \sigma_V|_{U\cap V}$, the desired value for the coboundary is $\delta\sigma = 0$. In other words, we calculate the difference of these two sections over the intersection.

Here we digress a little bit to say that there is a deep reason for wanting this particular intuitive meaning. Namely, the fact that among the original motivations for the creation of sheaf cohomology was the Mittag-Leffler problem of defining a meromorphic function with prescribed poles and principal parts. This is a problem of writing a global function with preassigned data at some points. If one has two local solutions defined on U and V, then they form a solution over $U \cup V$ provided they agree on the intersection $U \cap V$. The simplest case of course is when no poles are assigned and then we have the more familiar problem of analytic continuation. In general the n^{th} sheaf cohomology group will vanish when sections defined over $n+1$ open sets can be extended to n open sets. Otherwise, there will be obstructions to such extensions.

Coming back from the digression, we put $(\delta\sigma)_{U,V} = -\sigma_U + \sigma_V$. One than chooses the most natural and consistent generalization for higher order cochains. For a 1-cochain $\sigma = \{\sigma_{U,V}\}$ the coboundary $\delta\sigma$ is defined over triple intersections as $(\delta\sigma)_{U,V,W} = \sigma_{UV} - \sigma_{UW} + \sigma_{VW}$.

Definition: We define the *coboundary operator*

$$\delta : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F})$$

by the formula

$$(\delta\sigma)_{i_0,\cdots,i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j \sigma_{i_0,\cdots,\widehat{i_j},\cdots,i_{n+1}}\Big|_{U_{i_0}\cap\cdots\cap U_{i_{n+1}}}.$$

Exercise 7. (It is impossible to understand cohomology without at least once solving this exercise.) Show that $\delta^2 = 0$.

Definition: An n-cochain σ is a called a *cocycle* if $\delta\sigma = 0$. An n-cochain σ is called a *coboundary* if $\sigma = \delta\tau$ for some $n-1$ cochain τ.

Exercise $7'$. Show that every coboundary is a cocycle.

We are now ready to define the cohomology groups as the quotient of cocycles by coboundaries.

Definition: We set $Z^n(\mathcal{U}, \mathcal{F}) = Ker\,\delta \subset C^n(\mathcal{U}, \mathcal{F})$ and define the n^{th} *cohomology group* $H^n(\mathcal{U}, \mathcal{F})$ by the fomula

$$H^n(\mathcal{U}, \mathcal{F}) = \frac{Z^n(\mathcal{U}, \mathcal{F})}{\delta C^{n-1}(\mathcal{U}, \mathcal{F})}.$$

Given a sheaf $\mathcal{F}$ over a manifold M and a locally finite open cover of M we can now calculate the cohomology groups $H^n(\mathcal{U}, \mathcal{F})$. But these cohomology groups in principle depend on the open cover we chose. What we want is to calculate cohomologies of M itself. To get a cohomology for M we take a limit of cohomologies $\check{H}^n(\mathcal{U}, \mathcal{F})$ as the covering $\mathcal{U}$ becomes finer and finer. This limit is then the n^{th} Čech cohomology group and depends only on M and on $\mathcal{F}$ as we wish to have.

Definition: Given two locally finite open covers $\mathcal{U}$ and $\mathcal{U}'$ of M we say that $\mathcal{U}'$ is *finer* then $\mathcal{U}$ if every open set of $\mathcal{U}'$ is contained in some open set of $\mathcal{U}$, we write $\mathcal{U}' < \mathcal{U}$.

Exercise 8. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ and $\mathcal{U}' = \{U'_\alpha\}_{\alpha \in I'}$ and suppose $\mathcal{U}' < \mathcal{U}$. Show that we can choose a map $\phi : I' \to I$, such that
ι) $U'_\alpha \subset U_{\phi\alpha}$ for all α
$\iota\iota$) ϕ induces cochain maps $C^n(\mathcal{U}, \mathcal{F}) \to C^n(\mathcal{U}', \mathcal{F})$
$\iota\iota\iota$) ϕ induces cohomology maps $H^n(\mathcal{U}, \mathcal{F}) \to H^n(\mathcal{U}', \mathcal{F})$.

Definition: We define the n^{th} *Čech cohomology group* of $\mathcal{F}$ on M to be the direct limit of the $H^n(\mathcal{U}, \mathcal{F})$ as $\mathcal{U}$ becomes finer and finer:

$$\check{H}^n(M, \mathcal{F}) = \varinjlim H^n(\mathcal{U}, \mathcal{F}).$$

In practice, however, the direct limit is complicated to work with. Instead, what is used is the fact that under some condition on the cover $\mathcal{U}$ the equalities $H^n(\mathcal{U}, \mathcal{F}) = \check{H}^n(M, \mathcal{F})$ hold for all n. This condition is that $\mathcal{U}$ be acyclic for $\mathcal{F}$, in the sense that $H^n(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{F}) = 0$, for $n > 0$ and any $i_1, \cdots, i_p$, which is not difficult to obtain. In general one chooses covers whose intersections are open subsets of $\mathbf{R}^n$ (compare with exercise 10).

Exercise 9. Show that for any locally finite open cover $\mathcal{U}$ of M

$$\check{H}^0(M, \mathcal{F}) = H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(M).$$

Exercise 10. Show that $\check{H}^n(\mathbf{C}^q, \mathcal{O}) = 0$ for $n > 0$. Hint: Use the $\bar{\partial}$-Poincaré lemma.

Example: Calculations of $\check{H}^n(\mathbf{P}^1, \mathcal{O})$.

We first need an open cover for $\mathbf{P^1}$. Recall that topologically $\mathbf{P^1} \sim S^2$ and that S^2 has the canonical charts given by the stereographic projections from the South and the North poles. Our charts are $U = \{z\}$ and $V = \{\xi\}$ with $U \sim V \sim \mathbf{C}$ and $U \cap V \sim \mathbf{C} - \{0\}$ with $\xi = z^{-1}$ on $U \cap V$. The careful reader should verify that this cover is acyclic.

The cochains for this cover are:
$C^0(\{U, V\}, \mathcal{O}) = \{(f, g) : f \in \mathcal{O}(U), g \in \mathcal{O}(V)\}$
$C^1(\{U, V\}, \mathcal{O}) = \{h \in \mathcal{O}(U \cap V)\}$
$C^2(\{U, V\}, \mathcal{O}) = \emptyset$, since there are no triple intersections and by the same reasoning $C^n(\{U, V\}, \mathcal{O}) = \emptyset$, for $n \geq 2$.
It follows immediately that $\check{H}^n(\mathbf{P}^1, \mathcal{O}) = 0$ for $n \geq 2$.

Calculation of $\check{H}^0(\mathbf{P}^1, \mathcal{O})$: Given $(f, g) \in C^0(\{U, V\}, \mathcal{O})$ we have the power series expansions

$$f = \sum_{i=0}^{\infty} f_i u^i \quad g = \sum_{i=0}^{\infty} g_i \xi^i = \sum_{i=0}^{\infty} g_i z^{-i}.$$

Hence the coboundary $\delta((f, g)) = -f + g \in \mathcal{O}(U \cap V)$ is zero if and only if $f_i = g_i = 0$ for positive i and $f_0 = g_0$. It follows that the only 0-cocycles are constant global functions and $\check{H}^0(\mathbf{P}^1, \mathcal{O}) = \mathbf{C}$.

Calculation of $\check{H}^1(\mathbf{P}^1, \mathcal{O})$: First note that every 1-cochain is trivialy a 1-cocycle, because there are no triple intersections. A 1-cocycle $h \in C^1(\{U, V\}, \mathcal{O})$ has a power expansion of the form

$$h = \sum_{i=-\infty}^{\infty} h_i z^i.$$

Now we write h as a coboundary by putting

$$h = \sum_{i=0}^{\infty} h_i z^i + \sum_{i=-\infty}^{-1} h_i z^i = -f + g$$

where $f = \sum_{i=0}^{\infty} h_i z^i \in \mathcal{O}(U)$ and $g = \sum_{i=-\infty}^{-1} h_i z^i = \sum_{i=1}^{\infty} h_i \xi^i \in \mathcal{O}(V)$. We conclude that $\check{H}^1(\mathbf{P}^1, \mathcal{O}) = 0$.

Exercise 11. Use exercise 9 to show that $\check{H}^0(M, \mathcal{O}) = \mathbf{C}$ for any compact connected complex manifold. Hint: Use the maximum principle.

Example: Consider the complex line bundle over $\mathbf{P}^1$ given by transition matrix z^{-n} and denote the corresponding sheaf of local sections by $\mathcal{O}(n)$. We show that $\check{H}^0(\mathbf{P}^1, \mathcal{O}(2)) = \mathbf{C}^3$ and that $\check{H}^0(\mathbf{P}^1, \mathcal{O}(-2)) = 0$.

Calculation of $\check{H}^0(\mathbf{P}^1, \mathcal{O}(-2))$: We know from exercise 9 that $\check{H}^0(\mathbf{P}^1, \mathcal{O}(-2))$ consists of the global sections of $\mathcal{O}(-2)$. So we need to find global holomorphic sections for the bundle on $\mathbf{P}^1$ given by transition matrix (z^2). We use the same charts as in the previous example: $U = \{z\}$ and $V = \{\xi\}$ with $\xi = z^{-1}$ in $U \cap V$.

Suppose that we have a section f defined over U, then we may write $f = \sum_{i=0}^{\infty} a_i z^i$. If we change coordinates, then the section transforms to $(z^2)f = z^2 \sum_{i=0}^{\infty} a_i z^i$, which is never a holomorphic function on z^{-1} (unless it is zero) since it has only positive powers of z. Therefore, no (nonzero) section over U can be extended to V and we conclude that $\check{H}^0(\mathbf{P}^1, \mathcal{O}(-2)) = 0$.

Calculation of $\check{H}^0(\mathbf{P}^1, \mathcal{O}(2))$: Using the same charts as above, we look for a global section of the bundle given by transition matrix (z^{-2}). Just as before, a section defined over U has the form $f = \sum_{i=0}^{\infty} a_i z^i$ and under change of coordinates becomes $g = (z^{-2})f = z^{-2} \sum_{i=0}^{\infty} a_i z^i$. The function g is holomorphic in z^{-1} whenever $a_i = 0$ for $i > 2$, in which case it has the form: $g = a_0 z^{-2} + a_1 z^{-1} + a_2$. Therefore the global sections of $\mathcal{O}(2)$ depend on 3 complex parameters, and we conclude that $\check{H}^0(\mathbf{P}^1, \mathcal{O}(2)) = \mathbf{C}^3$.

Exercise 12. Show that

$$\check{H}^0(\mathbf{P}^1, \mathcal{O}(n)) = \begin{cases} 0 & if \quad n < 0 \\ \mathbf{C}^{n+1} & if \quad n \geq 0 \end{cases}.$$

III. SOME USEFUL RESULTS.

To end this lecture we mention (without proofs) some important results about how Čech cohomology compares with other cohomology theories.

Theorem III.1 *Let $H^*(M, \mathbf{Z})$ denote the simplicial cohomology of a manifold M with integer coefficients. Then*

$$H^*(M, \mathbf{Z}) = \check{H}^*(M, \mathbf{Z}).$$

Theorem III.2 *Let $H^*(M, \mathbf{R})$ denote the simplicial cohomology of a manifold M with real coefficients, $H^*_{sing}(M, \mathbf{R})$ denote the singular cohomology of M and $H^*_{DR}(M)$ de note the De Rham cohomology of M. Then*

$$H^*(M, \mathbf{R}) = H^*_{sing}(M, \mathbf{R}) = H^*_{DR}(M) = \check{H}^*(M, \mathbf{R}).$$

Theorem III.3 *Let $H^{p,q}_{\bar{\partial}}(M)$ denote Dolbeault cohomology of a complex manifold M and let Ω^p denote the sheaf of holomorphic p-forms on M. Then*

$$H^{p,q}_{\bar{\partial}}(M) = \check{H}^q(M, \Omega^p).$$

Using Hodge theory one proves that for a compact manifold M the Doulbeaut cohomology groups are finite dimensional. It then follows also that the cohomology groups $\check{H}^n(M, \mathcal{O})$ of a compact complex manifold M are finite dimensional.

Exercise 12. Show that $dim\, \check{H}^1(\mathbf{C}^2 - \{0\}, \mathcal{O}) = \infty$.

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[1] P. Griffths and J. Harris *principles of Algebraic Geometry, John Wiley & Sons Inc., N.Y. (1978)*
[2] Hartshorne, R.*Algebraic Geometry. Graduate Texts in Mathematics 56, Springer Verlag (1977)*