SELF-DUALITY FOR LANDAU–GINZBURG MODELS

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Abstract. In [Cl] Clarke describes mirror symmetry as a duality between Landau–Ginzburg models, so that the dual of an LG model (X, W) is another LG model $(X^\vee, W^\vee)$. We describe the examples where X is the total space of a vector bundle on $\mathbb{P}^1$ and we show that self-duality occurs in precisely two cases: the cotangent bundle and the resolved conifold.

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EG was supported by Fapesp grant 2012/10179-5 and RJ was supported by Faepex-PRP-Unicamp under grant number PRP/FAEPEX 005/2014 and Fapesp grant 2013/17654-3.

1. Introduction

For us a Landau–Ginzburg model (LG) is a variety X together with a regular function $W\colon X \to \mathbb{C}$ called the superpotential. Clarke [Cl] showed that one can state a generalized version of the Homological Mirror Symmetry conjecture of Kontsevich [Ko] as a duality between LG models. He also showed that this correspondence generalises those of Batyrev–Borisov Berglung–Hübsch, Givental, and Hori–Vafa.

This paper is an exercise in understanding the details of this correspondence. We summarise the construction in [Cl], which, for a given LG model (X, W), produced a dual $(X^\vee, W^\vee)$. When $(X^\vee, W^\vee) = (X, W)$ we call X self-dual. We then study the case when X is the total space of a vector bundle on $\mathbb{P}^1$ with an additional hypothesis. We prove that self-duality happens only in two cases: $X = \mathrm{Tot}(\mathcal{O}(-2))$ and $X = \mathrm{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$.

2. Acknowledgements

We thank Patrick Clarke for patiently explaining details of his work, and Tony Pantev for enlightening discussions.

3. The character to divisor map

Let X be a toric variety of rank n with a torus embedding $\iota\colon T \longrightarrow X$. The torus $T = (\mathbb{C}^*)^n$ is naturally an algebraic group; its algebraic functions are characters, that is, group morphisms, $\chi\colon T \longrightarrow \mathbb{C}^*$. Let M denote the group of characters of T and N the group of one-parameter subgroups, naturally identified with the dual of M, $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Let $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ denote the tensor products $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{R}} \mathbb{Z}$ respectively.

Since $\iota(T)$ is dense inside X, each character, $\chi \in M$, can be thought of as a rational map, $f_\chi\colon X \dashrightarrow \mathbb{C}$, which is nowhere zero on $\iota(T)$. Let $R = \{D_1, \ldots, D_r\}$ denote the set of irreducible components of $X \setminus \iota(T)$. These are prime T-invariant Weil divisors and can be read off the moment polytope for X. Because each $D \in R$ is irreducible and X is normal, one can compute the order of vanishing, $\mathrm{ord}_D(f_\chi)$, of f_χ along D. This defines a map,

$$\mathbf{div}(X)\colon M \longrightarrow \mathbb{Z}^R; \qquad \chi \mapsto (\mathrm{ord}_{D_1}(f_\chi), \ldots, \mathrm{ord}_{D_r}(f_\chi)).$$

Choosing ordered generators for M and an ordering of R gives a matrix $M_{\mathbf{div}}(X) \in \mathrm{Mat}_{n\times r}(\mathbb{Z})$. For each $D_r \in R$, let $v_r \in N$ be a generator for the corresponding ray in the fan. By [Ful, Section 3.3], $\mathrm{ord}_{D_r}(f_\chi) = \langle \chi, v_r \rangle$. This implies that, when the bases of N and M are dual, the rows of $\mathbf{div}_X$ can be calculated simply as generating vectors along the rays of the fan of X.

The cokernel of $\mathbf{div}(X)$ is the **Chow group** of X, written $A_{n-1}(X)$. When X is a complete toric variety the Chow group can be identified with the second integral cohomology $H^2(X, \mathbb{Z})$ and is torsion free. The following lemma is from [Cl].

Lemma 3.1. [Cl, Cor. 4.5] *If $D_1, \ldots, D_c$ are T-invariant Cartier divisors and X is the total space of the split bundle $\mathcal{O}_Y(-D_1) \oplus \cdots \oplus \mathcal{O}_Y(-D_c)$ over a toric variety Y, then the character group of X decomposes as*

$$M_X \cong M_Y \oplus \mathbb{Z}\sigma_1 \oplus \cdots \oplus \mathbb{Z}\sigma_c,$$

where σ_j is a rational section of $\mathcal{O}_Y(D_j)$ whose divisor is D_j, interpreted here as a character of T. The T-invariant Weil divisors of X are the preimages under p of the T-invariant Weil *divisors of Y as well as the total spaces X_j of the c subbundles $E_j^\vee$, where $E_j^\vee$ is the dual bundle to $\ker(\pi_j\colon E \to \mathcal{O}(D_j))$. Furthermore,*

$$\mathbf{div}_X = \left(\begin{array}{c|c|c|c} \mathbf{div}_Y & D_1 & \cdots & D_c \\ 0 & & \mathrm{id} & \end{array} \right).$$

with respect to the decomposition of M_X above and $\mathbb{Z}^{R_X} = \mathbb{Z}^{R_Y} \oplus \mathbb{Z}X_1 \oplus \cdots \oplus \mathbb{Z}X_c$.

4. The infinitesimal action on monomials

Let E be a vector bundle on a Kähler manifold Y with a global section $w \in H^0(Y, E)$. Assume that $X = \mathrm{Tot}(E^\vee)$ is a toric variety. A **superpotential** $W\colon X \to \mathbb{C}$ is a regular function on X. It can be determined by w as follows. In the category of coherent $\mathcal{O}_Y$-modules there are isomorphisms

$$H^0(Y, E) \cong \mathrm{Hom}(\mathcal{O}_Y, E) \cong \mathrm{Hom}(E^\vee, \mathcal{O}_Y).$$

Thus, w determines a morphism from $E^\vee$ to $\mathcal{O}_Y$, or, equivalently, a regular function the total space of $E^\vee$. Since T acts freely on the embedded torus $\iota(T) \subset X$, the zeroes of the function W must lie on the locus of T-invariant divisors. Thus, $W \circ \iota\colon T \to \mathbb{C}^*$ is a homomorphism of algebraic groups, which may be expressed as a finite linear sum of characters of T:

$$\iota^* W = \sum_{i=1}^{s} a_i \xi_i,$$

for scalars $a_i \in \mathbb{C}$ and characters $\xi_i \in M$. Set $\Xi := \{\xi_1, \ldots, \xi_s\}$.

The scalars $\{a_1, \ldots, a_s\}$ depend on the initial choice of embedding ι. In turn, the map ι is determined by a point $x \in X$, namely, the image of $1 \in T$. Write ι_x for the map sending $1 \mapsto x$. Then, if $x' = tx$ is another point in $\iota(T)$ for some $t \in T$, we have

$$\iota_{x'}^* W = \sum_{i=1}^{s} a_i \xi_i(t) \xi_i.$$

Let $(\mathbb{C}^*)^\Xi$ denote the space of all $\mathbb{C}^*$-linear sums of monomials in Ξ; these are functions on T. Now T acts on $(\mathbb{C}^*)^\Xi$ as above; that is, if $\iota_x^* W \in (\mathbb{C}^*)^\Xi$ and $t \in T$ then $t \cdot \iota_x^* W := \iota_{t\cdot x}^* W$. In order to eliminate the dependence of $\iota^* W$ on the choice of embedding, we consider $\iota^* W$ as an element of the quotient $(\mathbb{C}^*)^\Xi / T$. The kernel of the exponential map $\mathbb{C}^n \longrightarrow T; \quad (t_1, \ldots, t_n) \mapsto (e^{t_1}, \ldots, e^{t_n})$ is isomorphic to $\mathbb{Z}^n$, as is the lattice of one-parameter subgroups N. Let $\mathbb{Z}^\Xi$ denote the kernel of the corresponding exponential map on $\mathbb{C}^\Xi$. The action of T on $(\mathbb{C}^*)^\Xi$ gives a map $f\colon T \longrightarrow (\mathbb{C}^*)^\Xi; \quad t \mapsto t \cdot (\xi_1 + \cdots + \xi_s)$. Restricting the derivative $df\colon \mathbb{C}^r \longrightarrow \mathbb{C}^\Xi$ to the kernel N of $e^{(-)}$ gives a map which we denote by

$$\mathbf{mon}\colon N \longrightarrow \mathbb{Z}^\Xi.$$

Hence, the maps f, df, and **mon** fit between the following short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "\mathbf{mon}"] & \mathbb{C}^n \arrow[r, "e^{(-)}"] \arrow[d, "df"] & T \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & \mathbb{Z}^\Xi \arrow[r] & \mathbb{C}^\Xi \arrow[r, "e^{(-)}"] & (\mathbb{C}^*)^\Xi \arrow[r] & 0 \end{tikzcd}$$

Choosing and ordered basis for N and an ordering of the monomials in Ξ allows us express the map **mon** as a matrix $M_{\mathbf{mon}}(X) \in \mathrm{Mat}_{n\times s}(\mathbb{Z})$ such that the k^{th} row of this matrix is given by the integers $a_1, \dots, a_n$ defined by the equation $\xi_k(t_1, \dots, t_n) = t_1^{a_1} \cdots t_n^{a_n}$.

5. Toric LG models

A **toric Landau–Ginzburg model** is a triple, (X, W, K), where X is a toric variety, W is a regular function on X and $K \in A_{n-1}(X) \otimes_{\mathbb{Z}} \mathbb{C}/\mathbb{Z}$ is an element of the Chow group (with $\mathbb{C}/\mathbb{Z}$ coefficients). To such a model we have associated linear maps $\mathbf{div}(X)$ and $\mathbf{mon}(X)$. Choosing an element $L \in \operatorname{coker}(\mathbf{mon}) \otimes_{\mathbb{Z}} \mathbb{C}/\mathbb{Z}$ determines the **linear data** associated to (X, W, K), these are pairs $(\mathbf{div}, K)$ and $(\mathbf{mon}, L)$. We now provide an inverse to this construction.

First we specify conditions on $\mathbb{R}$-linear data (C, c) for it to yield an appropriate toric variety. Let $C \colon M \to \mathbb{Z}^r$ be a linear map, and $c \in \mathbb{Z}^r$. We say that the $\mathbb{R}$-linear data (C, c) is **kopaseptic** if

(1) the polyhedral set $P = \{ \xi \in M \mid C\xi + c \geq 0 \}$ associated to (C, c) has non-empty interior; and
(2) there exists a surjection $k \colon \mathbb{Z}^r \to \mathbb{Z}^{R_{X(C,c)}}$ sending standard generators to standard generators or zero such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "C"] \arrow[dr, "\mathbf{div}_{X(C,c)}"'] & \mathbb{Z}^r \arrow[d, "k"] \\ & \mathbb{Z}^{R_{X(C,c)}} \end{tikzcd},$$

where $R_{X(C,c)}$ denotes the number of torus-invariant divisors of the toric variety $X(C, c)$.

Condition 1 guarantees that the toric variety $X(C, c)$ corresponding to the polyhedral set of (C, c) is well-defined, and thus allows us to make sense of condition 2. Some of the inequalities $C\xi + c \geq 0$ defining the polyhedral set may be redundant and condition 2 tells us how to remove these redundances. In fact, k is almost uniquely determined, the only choice being which redundant condition to drop.

Now we need to determine precisely when a potential W (defined on a toric variety X) is regular. Since it is regular if and only if all its monomials are regular, and the **mon** matrix encodes all the information about those monomials, we can state our condition in terms of that matrix. Indeed, a monomial ξ is regular if and only if $\mathbf{div}\, \xi \geq 0$, so we have proved:

Lemma 5.1. *W is regular if and only if* $\mathbf{div} \circ \mathbf{mon}^T \geq 0$.

We now combine the above remarks into one definition. Let A and B be homomorphisms of free abelian groups of finite rank such that the domains of A and B have the same rank, and let K and L be elements in $\operatorname{coker}(A) \otimes_{\mathbb{Z}} \mathbb{C}/\mathbb{Z}$ and $\operatorname{coker}(B) \otimes_{\mathbb{Z}} \mathbb{C}/\mathbb{Z}$, respectively. A pair (A, K) and (B, L) is called a **$\mathbb{C}/\mathbb{Z}$-linear data**. Such data is said to be **kopaseptic** if

(1) $(A, \Im\mathfrak{m}(K))$ is kopaseptic; and
(2) the entries of the matrix $A \circ B^T$ are all non-negative.

Here $\Im\mathfrak{m} K$ denotes the imaginary part of K.

Given kopaseptic $\mathbb{C}/\mathbb{Z}$-linear data (A, K), (B, L), we can define the corresponding toric Landau–Ginzburg model (X, W, K) given by

(1) the toric variety $X := X(A, \Im\mathfrak{m}K)$ determined by A and $\Im\mathfrak{m}K$;
(2) the regular function $W := W(B, L)$ determined by B and L.

The element K specifies a choice of complexified Kähler class for our Landau–Ginzburg model.

6. Self-duality

Let (X, W, K) be a toric Landau–Ginzburg model with linear data $(\mathbf{div}(X), K)$, $(\mathbf{mon}, L)$. Then the **dual** $(X^\vee, W^\vee, K^\vee)$ of (X, W, K) is the toric Landau–Ginzburg model corresponding to the linear data obtained exchanging $(\mathbf{div}, K)$ and $(\mathbf{mon}, L)$.

Lemma 6.1. *Let (X, W, K) and (Y, W', K') be toric Landau–Ginzburg models. Then $(X \times Y, W + W', K + K')$ is a toric Landau–Ginzburg model and $\mathbf{div}(X \times Y) = \mathbf{div}(X) \oplus \mathbf{div}(Y)$ and $\mathbf{mon}(X \times Y) = \mathbf{mon}(X) \oplus \mathbf{mon}(Y)$.*

Proof. This follows directly from the definitions, given that the torus action on $X \times Y$ agrees with the original actions on X and Y. □

This immediately implies the following.

Corollary 6.2. *Suppose (X, W, K) is a toric Landau–Ginzburg model which is dual to $(X^\vee, W', K')$. Then $(X \times X^\vee, W + W', K + K')$ is self-dual.*

6.1. **The CY condition.** There are several inequivalent definitions of a Calabi–Yau manifold. Some authors require that the manifold be a compact complex Kähler manifold with a Ricci flat metric, while others use a stronger condition that implies the former: a compact complex Kähler manifold with trivial canonical bundle. When a Kähler manifold is non-compact then triviality of the canonical bundle does not necessarily imply the existence of a complete Ricci flat metric. In this case we make the following definition.

Definition 6.3. A complex Kähler manifold is **Calabi–Yau** if it has trivial canonical bundle and admits a complete Ricci-flat metric. This is called a **Calabi–Yau metric**.

The dual of a Calabi–Yau variety is expected to also be Calabi–Yau.

7. Self-duality for bundles on $\mathbb{P}^1$

We now describe such dualities for the case when our variety X is the total space of a vector bundle on $\mathbb{P}^1$.

7.1. **Rank 1.** Let $X = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$. For $k < 0$, E has no global sections, so assume $k \geq 0$. Choosing the chart $U := \{[z : 1] \mid z \in \mathbb{C}\}$ of $\mathbb{P}^1$ gives a chart on X on which points may be described as pairs (z, u), where u is a coordinate for the fibre of $E^\vee|_U$. Choosing the point $x = (1, 1)$ gives an embedding ι_x, so that an element $(t_1, t_2) \in T$ acts on X by $(t_1, t_2) \cdot (z, u) = (t_1 z, t_2 u)$. Having embedded the torus this way, Laurent polynomials in t_1 and t_2 can be interpreted as characters of the torus T and as rational functions on X. This gives a basis for the group of characters $M = \langle t_1, t_2 \rangle$; let v_1, v_2 be a dual basis for the one-parameter subgroups N. The T-invariant divisors of X are $f_0 = \{t_1 = 0\}$, $f_\infty = \{t_1 = \infty\}$ and $l = \{t_2 = 0\}$. The moment polytope for X is given by connecting the vertices $(0, 1)$-$(0, 0)$-$(1, 0)$-$(k+1, 1)$. Fig. 7.1 illustrates the case $k = 2$.

Remark 7.1. The unique value of k for which X is Calabi–Yau is $k = 2$.

Proposition 7.2. *The toric variety* $X = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ *belongs to a self-dual Landau–Ginzburg model* (X, W, K) *if and only if* $k = 2$.

Proof. With respect to the fixed basis above, the rows of the **div**-matrix are given by the vectors normal to the edges of the moment polytope, which are $(1, 0)$, $(0, 1)$, and $(-1, k)$. Hence

$$M_{\mathbf{div}}(X) = \begin{pmatrix} 1 & 0 \\ -1 & k \\ 0 & 1 \end{pmatrix}.$$

A global section w of E is represented by a polynomial of degree k (we assume $k \geq 0$). Identifying $\mathbb{P}^1$ with the subvariety of X cut out by $t_2 = 0$ gives a superpotential $W = a_0 t_2 + a_1 t_1 t_2 + \cdots + a_k t_1^k t_2$ for some $a_0, \ldots, a_k \in \mathbb{C}$. For X to belong to a self-dual toric Landau–Ginzburg model, there must exist a choice of basis for N and an ordering of Ξ such that $M_{\mathbf{div}}(X) = M_{\mathbf{mon}}(X)$. Clearly, Ξ must have cardinality three, so Ξ is a subset of three of the monomials in $\{t_2, \ldots, t_1^k t_2\}$. Choosing the basis for N, dual to M, a **mon**-matrix for X is given by

$$M_{\mathbf{mon}}(X) = \begin{pmatrix} a & 1 \\ b & 1 \\ c & 1 \end{pmatrix},$$

where a, b, c are distinct integers in $\{0, \ldots, k\}$. If a choice of basis for N exists such that $M_{\mathbf{mon}}(X) = M_{\mathbf{div}}(X)$, then there are (non-zero) integers $\lambda, \mu \in \mathbb{Z}$ such that $\lambda(a, b, c) + \mu(1, 1, 1) = (1, -1, 0)$. This gives $a + b - 2c = 0$. Likewise, there exist (non-zero) integers $\lambda', \mu' \in \mathbb{Z}$ such that $\lambda'(a, b, c) + \mu(1, 1, 1) = (0, k, 1)$. This implies $(k-1)a + b - kc = 0$. Together these two equations give $(k-2)(a-c) = 0$, which, since a and c are distinct, implies that $k = 2$.

It remains to show that, for $k = 2$, an element $K \in A_{n-1} \otimes_{\mathbb{Z}} \mathbb{C}/\mathbb{Z}$ can be chosen so that $(\mathbf{div}, \mathfrak{Im}(K))$ is kopaseptic. The Chow group in this case is isomorphic to $\mathbb{Z}$ by an isomorphism sending the generator $(1, 1, -2)$ in the codomain of $M_{\mathbf{div}}(X)$ to $1 \in \mathbb{Z}$. The polyhedral set defined by choosing $t > 0 \in A_{n-1}$ has a non-empty interior and produces inward normals that give the fan for X, whilst for $t \leq 0$, the relation from the third row of $M_{\mathbf{div}}(X)$ is made redundant. It follows that lifting $(1, 1, -2)$ to $\mathbb{C}/\mathbb{Z}$ gives a K such that (X, W, K) is self-dual. □

7.2. **Rank two bundles.** Now we consider the rank 2 bundles on $\mathbb{P}^1$ whose total space is Calabi–Yau, so $E = \mathcal{O}(-k) \oplus \mathcal{O}(k+2)$ on $Y = \mathbb{P}^1$. Let $X = W_k := \mathrm{Tot}\,(E^\vee)$. Note that $W_k \simeq W_{-k-2}$, so we can assume $k \geq -1$.

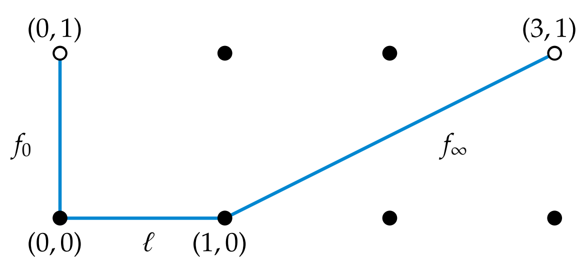

Figure 7.1 The moment polytope of $\mathrm{Tot}(\mathcal{O}(-2))$ with invariant divisors ℓ, f_0, and f_∞

Proposition 7.3. *The toric variety $X = W_k$ belongs to a self-dual toric Landau–Ginzburg model (X, W, K) if and only if $k = 0, -1$.*

Proof. As in the example above, choosing the chart $U := \{[z : 1] \mid z \in \mathbb{C}\}$ of $\mathbb{P}^1$ gives a chart on X on which points may be described as triples, (z, u, v), where u is a coordinate along a fibre of $\mathcal{O}(k)|_U$ and v is a coordinate along a fibre of $\mathcal{O}(-k-2)|_U$. Let $T = (\mathbb{C}^*)^3$ embedded in X so that $(t_1, t_2, t_3) \in T$ acts by the rule $(t_1, t_2, t_3) \cdot (z, u, v) = (t_1 z, t_2 u, t_3 v)$. Again, we let Laurent polynomials in t_i represent the characters of T and rational functions on X. With this notation, the T-invariant divisors are $f_0 = \{t_1 = 0\}$, $f_\infty = \{t_1 = \infty\}$, $l_1 = \{t_2 = 0\}$ and $l_2 = \{t_3 = 0\}$. Let $[\infty]$ denote the divisor of $\mathbb{P}^1$ which is the intersection of $\mathbb{P}^1$ with f_∞ in X. Applying Lemma 3.1 with $c = 2$, $D_1 = -k[\infty]$, $D_2 = (k+2)[\infty]$, $\sigma_1 = t_2$ and $\sigma_2 = t_3$ gives the matrix

$$M_{\mathbf{div}}(X) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -k & k+2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The following three cases describe the global sections of $E = \mathcal{O}(-k) \oplus \mathcal{O}(2+k)$ on $\mathbb{P}^1$.

$$H^0(\mathbb{P}^1, E) = \begin{cases} H^0(\mathbb{P}^1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \cong \mathbb{C}[x]_1 \oplus \mathbb{C}[x]_1 & \text{when } k = -1, \\ H^0(\mathbb{P}^1, \mathcal{O}(2) \oplus \mathcal{O}) \cong \mathbb{C}[x]_2 \oplus \mathbb{C} & \text{when } k = 0, \\ H^0(\mathbb{P}^1, \mathcal{O}(k+2)) \cong \mathbb{C}[x]_{k+2} & \text{when } k \geq 1. \end{cases}$$

When $k \geq 0$ the **div** and **mon** matrices decompose into the direct sum of the **div** and **mon** matrices for $\mathrm{Tot}(\mathcal{O}(-k-2))$ with the identity matrix. That X belongs to a self-dual Landau–Ginzburg model for $k = 0$ but not for $k \geq 1$ follows from Proposition 7.2.

Consider $k = -1$. A generic section of E is a pair of linear polynomials in a single variable. This produces the superpotential $W = a_0 t_2 + a_1 t_1 t_2 + b_0 t_3 + b_1 t_1 t_3$ on X, where $a_0, a_1, b_0, b_1 \in \mathbb{C}$. Judiciously order the monomials in W so that $\Xi = \{t_1 t_3, t_2, t_3, t_1 t_2\}$. Let s_1, s_2, s_3 denote one-parameter subgroups dual to the characters t_1, t_2, t_3. Finally, choose the basis $N = \langle s_1 s_3, s_3, s_1 s_2 \rangle$. With respect to these choices $M_{\mathbf{mon}}(W_k) = M_{\mathbf{div}}(W_k)$.

The Chow group is isomorphic to $\mathbb{Z}$, the subgroup $\{(t, t, -t, -t) \mid t \in \mathbb{Z}\}$ of the codomain of $M_{\mathbf{div}}(X)$. Again, if $t < 0$ then the relations from the third and forth rows of $M_{\mathbf{div}}(X)$ are redundant, but $t > 0$ produces a polytope with inward normals which define the fan for X. Choosing a lift of $(1, 1, -1, -1)$ yields the required K. □

7.3. **Higher rank bundles.** Recall the following definition.

Definition 7.4. A vector bundle on a curve is **polystable** if it is isomorphic to a sum of stable bundles with the same slope.

The following theorem of Hori is from [Hori, Theorem 32.8.8].

Theorem 7.5. *(Hori) A holomorphic vector bundle admits a Calabi–Yau metric if and only if it is polystable.*

Theorem 7.6. *Let X be the total space of a vector bundle on $\mathbb{P}^1$. Suppose, additionally, that such a bundle is Calabi–Yau. Then X is self-dual if and only if $X = \mathcal{O}(-2)$ or $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.*

Proof. The previous sections deal with the rank one and two cases. Grothendieck splitting lemma says that a rank n bundle E on $\mathbb{P}^1$ splits as a sum of line bundles $E \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$. The total space of E has trivial canonical bundle if and only if $\sum a_i = -2$.

If E is a sum of two line bundles, $\mathcal{O}(a) \oplus \mathcal{O}(b)$, with $a \geq b$ then the slope of $\mathcal{O}(a)$ is greater than or equal to the slope of E. Induction on the rank r of E, for $r \geq 2$, shows that vector bundles on $\mathbb{P}^1$ of rank $r \geq 2$ are not stable. Thus, the only stable vector bundles on $\mathbb{P}^1$ are the line bundles. It follows that a vector bundle on $\mathbb{P}^1$ is polystable if and only if it is of the form

$$\mathcal{O}(a) \oplus \cdots \oplus \mathcal{O}(a)$$

for some a.Therefore, vector bundles on $\mathbb{P}^1$ with rank greater than two do not satisfy the Calabi–Yau condition required for self-duality. □

Remark 7.7. We expect that the hypothesis that the bundle is Calabi–Yau can be removed from this theorem.

Remark 7.8. The Calabi–Yau condition used in Theorem 7.6 is stronger than the commonly used definition that only requires triviality of the canonical bundle. Using the latter, one can apply the algebraic argument from the proof of Proposition 7.2 to show that a Calabi–Yau vector bundle on $\mathbb{P}^1$ can also be a direct sum of $\mathcal{O}(-2)$ or $\mathcal{O}(-1)^{\oplus 2}$ with $\mathcal{O}^{\oplus k}$ for some $k \geq 0$. That the former, stronger condition removes the trivial summands gives justification of its suitability.

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